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Lattice-Algebraic Morphology

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Abstract

The relations between two abstract lattice-algebraical approaches to mathematical morphology are investigated. One approach, developed by Heijmans and Ronse, entails the use of an abelian automorphism group, G, acting transitively on a sup-generating subset of the lattice, in order to abstract the translation invariance present in concrete morphology theories. The other, developed by Banon and Barrera, analyzes general mappings between complete lattices and develops morphological decomposition formulas for such mappings. By determining the G-invariant forms of the concepts and theorems of the Banon-Barrera theory, the present investigation combines the two theories into a coherent whole and develops them further.

Contents

1 Introduction

This report details an investigation to develop mathematical tools for designing improved or more nearly optimal algorithms for automatic/aided target recognition (ATR) in systems employing such target sensors as synthetic aperture and laser radars (SARs and ladars), and forward-looking infrared sensors (FLIRs). The investigation specifically concerns the development of such tools within the field of digital image processing and analysis known as *mathematical morphology*, a field that emerged in the early sixties in the *Fontainebleau School* of Serra and Matheron as a bottom-up, hierarchical approach to image analysis. Mathematical morphology has since found numerous practitioners throughout Europe, the United States, and South America, has been successfully applied in such diverse fields as materials science, microscopic imaging, pattern recognition, medical imaging, and computer vision, and is today one of the principal systematic methodologies employed in image-recognition research and practice, including ATR.

In previous reports [1–3], I have reviewed and added to the theory that supports the use of Euclidean morphology in processing both binary and greyscale imagery, and have contributed a body of results that generalize the topological aspect of mathematical morphology to the realm of complete lattices that have an *upper continuity* property, which is dually related to the concept of a *continuous lattice* [4].

This report contributes results in line with the recent trend among mathematical morphology theorists to abstract the lattice-theoretical essentials of the theory, and, through such generalization, enhance the theory's power for systematic applications. Indeed, much of the recent work in mathematical morphology has been in developing it as an abstract lattice-algebraical theory. Prominent examples of this work can be found in Serra [5], Heijmans and Ronse [6], Heijmans [7], and Banon and Barrera [8].

In the last cited article, Banon and Barrera present a theory of arbitrary mappings between *complete lattices* that accomplishes the following:

1. It defines the elementary lattice mappings called *erosions*, *dilations*, *anti-erosions*, and *anti-dilations*, which in turn generalize the more specific and common morphological concepts [9] that go by the same names.

2. It defines and develops the concept of a *morphological connection*, which generalizes the concepts *adjunction* and *Galois connection* that Serra and Achache have used to fruitfully relate the elementary lattice mappings among themselves.

3. It finally culminates in a general decomposition theorem for complete lattice mappings in two alternative forms. In one the general mapping is decomposed as a supremum of certain so-called *sup-generating* mappings; in the other it is expressed as an infimum of certain so-called *inf-generating* mappings. Sup-generating mappings turn out to be expressable as the infimum of an erosion and an anti-dilation, while inf-generating mappings are expressable as the supremum of a dilation and an anti-erosion.

The Banon-Barrera article thus succeeds in developing pure lattice algebra along lines that suggest that mathematical morphology can be profitably viewed as essentially the general theory of mappings between complete lattices, or at least that the latter is a useful abstract perspective in which to view the various concrete forms of mathematical morphology (e.g., set morphology, function morphology, closed Euclidean set morphology, upper semicontinuous function morphology).

In the cited article of Heijmans and Ronse, and more fully in the book by Heijmans, a lattice theoretical morphology is developed based on the idea that the role played by translations in concrete morphologies can be generalized to a complete lattice $\mathcal{L}$ by introducing an appropriate group, G, of automorphisms of $\mathcal{L}$; more specifically, $\mathcal{L}$ is assumed to have a *sup-generating* subset ℓ (which means that every element x of $\mathcal{L}$ is the supremum of the elements of ℓ that preceed x), G is assumed to be an abelian group that *acts effectively* (as a group of automorphisms) on $\mathcal{L}$, and the action of G is assumed to be compatible with ℓ in the sense that G *acts transitively* on ℓ and ℓ is G*-invariant.* This approach produces a theory of G-invariant complete-lattice mappings; leads to abstract definitions of G-invariant erosions, dilations, etc; results in a decomposition theorem for G-invariant complete-lattice mappings similar to that of Banon and Barrera; and is intuitively closer to the more concrete morphology theories.

This report investigates the relations between these two approaches, and, by determining the G-invariant forms of the concepts and theorems of the Banon-Barrera theory, combines them into a coherent whole. Section 2 gives a summary exposition of the Banon-Barrera theory; the reader is forewarned, however, that this material is rife with technicalities, and that I present it without the proofs given in the cited Banon-Barrera article. My purpose in presenting this theory is not to give a didactic exposition, but to intelligibly set it down as succinctly as possible so that I can later recall and use it. I introduce the group theoretical approach of Heijmans and Ronse in section 3, and then set about, in sections 3.2 to 3.4—which contain my contributions to this research area—to develop this approach in light of the results of the Banon-Barrera theory, by determining the group-invariant form of the latter. Section 4 then gives three examples from standard mathematical morphology that illustrate the combined theory; several original concepts and results, needed to relate the examples to the theory, are contained in this section. Finally, in section 5, I present a conjecture that has an important bearing on the potential enrichment of the combined theory that could accrue from the additional structure possessed by $\mathbf{M}$*-topologized upper continuous lattices* [3].

For the theory and terminology of lattices and posets (partially ordered sets), I follow Birkhoff [10]. The definitions and facts below are given for reference.

(1) A **sup-lattice** is a poset in which every finite subset has a least upper bound or supremum. An **inf-lattice** is a poset in which every finite subset has a greatest lower bound or infimum. The **morphisms of sup-lattices (inf-lattices)** are the mappings of $\mathcal{L}$ into $\mathcal{M}$, where $\mathcal{L}$ and $\mathcal{M}$ are sup-lattices (inf-lattices), that preserve finite suprema (finite infima). Inf- and sup-lattices are referred to collectively as **semilattices**. An inf-lattice (sup-lattice) in which every nonempty subset has an infimum (supremum) is called **complete**. A lattice in which every nonempty subset has both an infimum

and a supremum is called a **complete lattice**. The morphisms of complete inf-lattices that preserve arbitrary infima are called **(meet-) complete**; similarly, the morphisms of complete sup-lattices that preserve arbitrary suprema are called **(join-) complete**. The homomorphisms of complete lattices that preserve arbitrary infima (suprema) are also called **meet-complete (join-complete)**, and those that preserve arbitrary infima and suprema are called **complete homomorphisms**.

(2) A one-to-one lattice homomorphism is a lattice isomorphism, a one-to-one semilattice morphism is a semilattice isomorphism, and the isomorphisms of complete semilattices and complete lattices are complete.

2 Mappings Between Complete Lattices

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete lattices and let $\mathcal{O}_{12} = \mathcal{L}_2^{\mathcal{L}_1}$ denote the set of mappings of $\mathcal{L}_1$ into $\mathcal{L}_2$. In the special case where $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$, we let $\mathcal{O} = \mathcal{O}(\mathcal{L}) = \mathcal{L}^{\mathcal{L}}$ and call the mappings in $\mathcal{O}$ **operators on** $\mathcal{L}$. We give $\mathcal{O}_{12}$ the **pointwise ordering** $\preceq_p$ by defining

$$\psi \preceq_p \varphi \iff \psi(x) \preceq \varphi(x) \text{ for all } x \in \mathcal{L}_1.$$

More generally, if L and M are lattices, then M^L is partially ordered by $\preceq_p$.

Remark 1 *If L and M are lattices and M is complete, then M^L is a complete lattice relative to $\preceq_p$. In particular, $\mathcal{O}_{12}$ is a complete lattice relative to $\preceq_p$; in fact, if $\mathcal{A} \subset \mathcal{O}_{12}$, then* $(\sup \mathcal{A})(x) = \sup\{\psi(x) : \psi \in \mathcal{A}\}$ *and* $(\inf \mathcal{A})(x) = \inf\{\psi(x) : \psi \in \mathcal{A}\}$.

The universal bounds $\mathbf{o}$ and $\mathbf{e}$ of $\mathcal{O}_{12}$ are the mappings defined for all $x \in \mathcal{L}_1$ by $\mathbf{o}(x) = o$ and $\mathbf{e}(x) = e$, where o and e are the universal bounds of $\mathcal{L}_2$. We denote the universal bounds of $\mathcal{L}_1$ by O and E. Thus $\mathbf{O}(y) = O$ and $\mathbf{E}(y) = E$, each for all $y \in \mathcal{L}_2$, define the universal bounds of $\mathcal{O}_{21}$. Next we define the concept of a *complete sublattice.*

Definition 1 *If M is a sublattice of a complete lattice L, then M is called a* **complete sublattice** *of L if M is complete and the following hold.*

1. $\inf_M B = \inf_L B \ \forall \ B \subset M.$
2. $\sup_M B = \sup_L B \ \forall \ B \subset M.$

M is called a **meet-complete** [**join-complete**] *sublattice of L if only* (1) [(2)] *holds. Thus M is a complete sublattice if and only if M is both meet-complete and join-complete.*

Remark 2 *If M is a sublattice of a complete lattice L, and if M is complete, then M is a complete sublattice of L if and only if $\inf_L B$ and $\sup_L B$ lie in M for every subset B of M.*

Remark 3 *If M is a sublattice of a complete lattice L, then M is a meet-complete (join-complete) sublattice of L if M has a universal upper (lower) bound and every subset B of M satisfies $\inf_L B \in M$ $(\sup_L B \in M)$.*

2.1 Increasing and Decreasing Mappings

Definition 2 *Let X and Y be posets and let $f : X \longrightarrow Y$.*

1. *f is called* **increasing** *if $x, x' \in X$ and $x \preceq x' \Longrightarrow f(x) \preceq f(x')$.*
2. *f is called* **decreasing** *if $x, x' \in X$ and $x \preceq x' \Longrightarrow f(x') \preceq f(x)$.*

Isotone and antitone are synonyms for increasing and decreasing.

We denote the increasing and decreasing mappings in $\mathcal{O}_{12}$ by $\mathcal{O}_{12}^{+}$ and $\mathcal{O}_{12}^{-}$, respectively.

Proposition 1 *$\mathcal{O}_{12}^{+}$ and $\mathcal{O}_{12}^{-}$ are complete sublattices of $\mathcal{O}_{12}$.*

Proposition 2 *If $\psi \in \mathcal{O}_{12}$, then, of the following six statements, the first three are equivalent and the last three are equivalent.*

1. *ψ is increasing.*
2. $\psi(\inf B) \preceq \inf \psi(B) \ \forall\ B \subset \mathcal{L}_1$.
3. $\sup \psi(B) \preceq \psi(\sup B) \ \forall\ B \subset \mathcal{L}_1$.
4. *ψ is decreasing.*
5. $\psi(\sup B) \preceq \inf \psi(B) \ \forall\ B \subset \mathcal{L}_1$.
6. $\sup \psi(B) \preceq \psi(\inf B) \ \forall\ B \subset \mathcal{L}_1$.

2.2 Erosions, Dilations, Anti-Erosions, and Anti-Dilations

Definition 3 *If $\psi \in \mathcal{O}_{12}$, then ψ is called*

1. *an* **erosion** *if ψ is a meet-complete inf-lattice morphism, i.e., if*
$$\psi(\inf B) = \inf \psi(B) \ \forall\ B \subset \mathcal{L}_1,$$
2. *a* **dilation** *if ψ is a join-complete sup-lattice morphism, i.e., if*
$$\psi(\sup B) = \sup \psi(B) \ \forall\ B \subset \mathcal{L}_1,$$
3. *an* **anti-erosion** *if* $\psi(\inf B) = \sup \psi(B) \ \forall\ B \subset \mathcal{L}_1$,
4. *an* **anti-dilation** *if* $\psi(\sup B) = \inf \psi(B) \ \forall\ B \subset \mathcal{L}_1$.

Let us adopt the suggestive notations $\mathcal{E}_{12}$, $\mathcal{D}_{12}$, $\widetilde{\mathcal{E}}_{12}$, and $\widetilde{\mathcal{D}}_{12}$ for the erosions, dilations, anti-erosions, and anti-dilations, respectively, in $\mathcal{O}_{12}$.

Proposition 3 *Then we have the following:*

1. *$\mathcal{E}_{12}$ and $\mathcal{D}_{12}$ are subsets of $\mathcal{O}_{12}^{+}$; $\widetilde{\mathcal{E}}_{12}$ and $\widetilde{\mathcal{D}}_{12}$ are subsets of $\mathcal{O}_{12}^{-}$.*
2. *$\mathcal{E}_{12}$, $\mathcal{D}_{12}$, $\widetilde{\mathcal{E}}_{12}$, and $\widetilde{\mathcal{D}}_{12}$ are complete lattices relative to $\preceq_p$.*
3. *$\mathcal{E}_{12}$, $\mathcal{D}_{12}$, $\widetilde{\mathcal{E}}_{12}$, and $\widetilde{\mathcal{D}}_{12}$ are not complete sublattices of $\mathcal{O}_{12}$.*
4. *$\mathcal{E}_{12}$ and $\widetilde{\mathcal{D}}_{12}$ are meet-complete sublattices of $\mathcal{O}_{12}$.*
5. *$\mathcal{D}_{12}$ and $\widetilde{\mathcal{E}}_{12}$ are join-complete sublattices of $\mathcal{O}_{12}$.*

Proof $\mathcal{E}_{12}$ and $\widetilde{\mathcal{D}}_{12}$ ($\mathcal{D}_{12}$ and $\widetilde{\mathcal{E}}_{12}$) are inf-closed (sup-closed), $\mathbf{e} \in \mathcal{E}_{12}$, $\mathbf{o} \in \mathcal{D}_{12}$, $\mathbf{o} \in \widetilde{\mathcal{E}}_{12}$, and $\mathbf{e} \in \widetilde{\mathcal{D}}_{12}$. In addition, $\inf_{\mathcal{E}_{12}} = \inf_{\mathcal{O}_{12}}$, $\sup_{\mathcal{D}_{12}} = \sup_{\mathcal{O}_{12}}$, $\inf_{\widetilde{\mathcal{D}}_{12}} = \inf_{\mathcal{O}_{12}}$, and $\sup_{\widetilde{\mathcal{E}}_{12}} = \sup_{\mathcal{O}_{12}}$; but $\sup_{\mathcal{E}_{12}} \neq \sup_{\mathcal{O}_{12}}$, $\inf_{\mathcal{D}_{12}} \neq \inf_{\mathcal{O}_{12}}$, $\sup_{\widetilde{\mathcal{D}}_{12}} \neq \sup_{\mathcal{O}_{12}}$, and $\inf_{\widetilde{\mathcal{E}}_{12}} \neq \inf_{\mathcal{O}_{12}}$.

2.3 Inf- and Sup-Separable Mappings

Definition 4 *Let X and Y be posets, and let $f : X \longrightarrow Y$.*

1. *f is called* **sup-separable** *if: Given $a, b, x \in X$ such that $a \preceq x \preceq b$, then*

$$f(x) \preceq y \text{ for all } y \in Y \text{ such that } f(a) \preceq y \text{ and } f(b) \preceq y.$$

2. *f is called* **inf-separable** *if: Given $a, b, x \in X$ such that $a \preceq x \preceq b$, then*

$$y \preceq f(x) \text{ for all } y \in Y \text{ such that } y \preceq f(a) \text{ and } y \preceq f(b).$$

Proposition 4 *If $\psi \in \mathcal{O}_{12}$, then ψ is*

1. *inf-separable if and only if $\psi(\inf B) \wedge \psi(\sup B) \preceq \inf \psi(B)\ \forall\ B \subset \mathcal{L}_1$,*
2. *sup-separable if and only if $\sup \psi(B) \preceq \psi(\inf B) \vee \psi(\sup B)\ \forall\ B \subset \mathcal{L}_1$.*

Remark 4 *Every $\psi \in \mathcal{O}_{12}^{+} \cup \mathcal{O}_{12}^{-}$ is both inf- and sup-separable.*

Proposition 5 *If L and M are lattices and M is complete, then the inf-separable (sup-separable) mappings of L into M form a complete lattice relative to $\preceq_p$. In particular, the inf-separable (sup-separable) mappings in $\mathcal{O}_{12}$ form a complete lattice relative to $\preceq_p$.*

2.4 Inf- and Sup-Generating Mappings

Definition 5 *If $\psi \in \mathcal{O}_{12}$, then ψ is said to be*

1. **inf-generating** *if $\sup \psi(B) = \psi(\inf B) \vee \psi(\sup B)$ for all nonempty $B \subset \mathcal{L}_1$,*
2. **sup-generating** *if $\psi(\inf B) \wedge \psi(\sup B) = \inf \psi(B)$ for all nonempty $B \subset \mathcal{L}_1$.*

Let $\mathbf{\Lambda}_{12}$ and $\mathbf{M}_{12}$ **denote the sup- and inf-generating mappings in** $\mathcal{O}_{12}$, respectively.

Proposition 6 *Then we have the following:*

1. *$\mathcal{E}_{12} \cup \widetilde{\mathcal{D}}_{12} \subset \mathbf{\Lambda}_{12}$ and $\mathcal{D}_{12} \cup \widetilde{\mathcal{E}}_{12} \subset \mathbf{M}_{12}$.*
2. *If $\psi \in \mathbf{\Lambda}_{12}$, then ψ is inf-separable; if $\psi \in \mathbf{M}_{12}$, then ψ is sup-separable.*
3. *$\mathbf{\Lambda}_{12}$ is a meet-complete sublattice of $\mathcal{O}_{12}$ and $\mathbf{M}_{12}$ is a join-complete sublattice of $\mathcal{O}_{12}$.*
4. *Neither $\mathbf{\Lambda}_{12}$ nor $\mathbf{M}_{12}$ is a complete sublattice of $\mathcal{O}_{12}$.*
5. *The universal bounds of both $\mathbf{\Lambda}_{12}$ and $\mathbf{M}_{12}$ are $\mathbf{o}$ and $\mathbf{e}$.*

2.5 Galois Connections and Adjunctions

Definition 6 *Let X and Y be posets, let $f : X \longrightarrow Y$, and let $g : Y \longrightarrow X$. If f and g are decreasing, then the pair (f, g) is called a* **Galois connection** *between X and Y if for all $x \in X$ and $y \in Y$ we have that $x \preceq (g \circ f)(x)$ and $y \preceq (f \circ g)(y)$.*

Proposition 7 *Let X and Y be posets, let $f : X \longrightarrow Y$, and let $g : Y \longrightarrow X$. Then (f, g) is a Galois connection if and only if $x \preceq g(y) \iff y \preceq f(x)\ \forall\ (x, y) \in X \times Y$.*

Definition 7 *An* **adjunction** *between X and Y is a Galois connection between $\widetilde{X}$ and Y; that is, if X and Y are posets, then a pair (f, g) of functions, $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$, is an adjunction between X and Y if and only if f and g are increasing and, for all $x \in X$ and $y \in Y$, we have that $(g \circ f)(x) \preceq x$ and $y \preceq (f \circ g)(y)$.*

In the above, $\widetilde{X}$ denotes the dual of the poset X—i.e., the set X with the reverse ordering. An adjunction (Galois connection) between X and X is said, more simply, to be on X.

Proposition 8 *Let X and Y be posets, let $f : X \longrightarrow Y$, and let $g : Y \longrightarrow X$. Then (f, g) is an adjunction between X and Y if and only if $g(y) \preceq x \iff y \preceq f(x)\ \forall\ (x, y) \in X \times Y$.*

Proposition 9 (Achache) *The set of Galois connections between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the graph of a lattice isomorphism between $\widetilde{\mathcal{D}}_{12}$ and $\widetilde{\mathcal{D}}_{21}$. More specifically: If $(\widetilde{\delta}, \widetilde{\varsigma})$ is a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$, then the following hold.*

(a) *$\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ and $\widetilde{\varsigma} \in \widetilde{\mathcal{D}}_{21}$.*

(b) *$\widetilde{\delta}(x) = \bigvee\{y \in \mathcal{L}_2 : x \preceq \widetilde{\varsigma}(y)\}\ \forall\ x \in \mathcal{L}_1$ and $\widetilde{\varsigma}(y) = \bigvee\{x \in \mathcal{L}_1 : y \preceq \widetilde{\delta}(x)\}\ \forall\ y \in \mathcal{L}_2$.*

Furthermore, to each $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ there corresponds a unique $\widetilde{\varsigma} \in \widetilde{\mathcal{D}}_{21}$ such that $(\widetilde{\delta}, \widetilde{\varsigma})$ is a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$; conversely, to each $\widetilde{\varsigma} \in \widetilde{\mathcal{D}}_{21}$ there corresponds a unique $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ such that $(\widetilde{\delta}, \widetilde{\varsigma})$ is a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$; and finally, the one-to-one correspondence thusly described between $\widetilde{\mathcal{D}}_{12}$ and $\widetilde{\mathcal{D}}_{21}$ is a lattice isomorphism.

Proposition 10 (Serra) *The set of adjunctions between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the graph of a dual-lattice isomorphism between $\mathcal{E}_{12}$ and $\mathcal{D}_{21}$. More specifically: If (ε, ς) is an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$, then the following hold.*

(a) *$\varepsilon \in \mathcal{E}_{12}$ and $\varsigma \in \mathcal{D}_{21}$*

(b) *$\varepsilon(x) = \bigvee\{y \in \mathcal{L}_2 : \varsigma(y) \preceq x\}\ \forall\ x \in \mathcal{L}_1$ and $\varsigma(y) = \bigwedge\{x \in \mathcal{L}_1 : y \preceq \varepsilon(x)\}\ \forall\ y \in \mathcal{L}_2$.*

Furthermore, to each $\varepsilon \in \mathcal{E}_{12}$ there corresponds a unique $\varsigma \in \mathcal{D}_{21}$ such that (ε, ς) is an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$; conversely, to each $\varsigma \in \mathcal{D}_{21}$ there corresponds a unique $\varepsilon \in \mathcal{E}_{12}$ such that (ε, ς) is an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$; and finally, the one-to-one correspondence thusly described between $\mathcal{E}_{12}$ and $\mathcal{D}_{21}$ is a dual-lattice isomorphism.

2.6 Morphological Connections

Definition 8 *If X and Y are posets, $g, h : Y \longrightarrow X$, and $f : X \longrightarrow Y$ is inf-separable, then $(f, (g, h))$ is called a* **morphological connection** *between X and Y if the following hold.*

1. *For all $y, y' \in Y$ such that $g(y') \preceq h(y')$, $y \preceq y' \Longrightarrow g(y) \preceq g(y')$ and $h(y') \preceq h(y)$.*
2. *$(g \circ f)(x) \preceq x$ and $x \preceq (h \circ f)(x)$ for all $x \in X$.*
3. *$y \preceq (f \circ g)(y)$ and $y \preceq (f \circ h)(y)$ for all $y \in Y$ such that $g(y) \preceq h(y)$.*

Proposition 11 *Let X and Y be posets, let $g, h : Y \longrightarrow X$, and let $f : X \longrightarrow Y$ be inf-separable. Then $(f, (g, h))$ is a morphological connection between X and Y if and only if*

$$g(y) \preceq x \preceq h(y) \iff y \preceq f(x) \; \forall \; (x, y) \in X \times Y.$$

Remark 5 *Let $\mathcal{L}_1$ be a complete lattice, let X be a poset, and let* $\mathbf{O}$ *and* $\mathbf{E}$ *denote the universal bounds of the complete lattice $\mathcal{L}_1^X$. Then we have the following:*

1. *$(\psi, (\mathbf{O}, \beta))$ is a morphological connection between $\mathcal{L}_1$ and X if and only if (ψ, β) is a Galois connection between $\mathcal{L}_1$ and X.*
2. *$(\psi, (\alpha, \mathbf{E}))$ is a morphological connection between $\mathcal{L}_1$ and X if and only if (ψ, α) is an adjunction between $\mathcal{L}_1$ and X.*

Recall that we are denoting the universal bounds of $\mathcal{L}_1$ by O and E.

Definition 9 *Let $\mathbf{\Delta}_{21}$ denote the set of pairs $(\alpha, \beta) \in (\mathcal{O}_{21} \times \mathcal{O}_{21})$ such that*

$$\alpha(y_0) \not\preceq \beta(y_0) \text{ for some } y_0 \in \mathcal{L}_2 \Longrightarrow \alpha(y) = E \text{ and } \beta(y) = O \text{ for all } y \in \mathcal{L}_2,$$

that is, such that either $\alpha \preceq_p \beta$ or $\alpha = \mathbf{E}$ and $\beta = \mathbf{O}$. Also, let $\mathcal{D}\widetilde{\mathcal{D}}_{21}$ and $\widetilde{\mathcal{E}}\mathcal{E}_{21}$, respectively, denote the sets $(\mathcal{D}_{21} \times \widetilde{\mathcal{D}}_{21}) \cap \mathbf{\Delta}_{21}$ and $(\widetilde{\mathcal{E}}_{21} \times \mathcal{E}_{21}) \cap \mathbf{\Delta}_{21}$.

Definition 10 *If $\psi \in \mathcal{O}_{12}$ and $\alpha, \beta \in \mathcal{O}_{21}$, then define the following related mappings.*

1. $\underline{\cdot\psi}(y) = \bigwedge\{x \in \mathcal{L}_1 : y \preceq \psi(x)\} \; \forall \; y \in \mathcal{L}_2$.
2. $\overline{\cdot\psi}(y) = \bigvee\{x \in \mathcal{L}_1 : y \preceq \psi(x)\} \; \forall \; y \in \mathcal{L}_2$.
3. $\underline{\psi\cdot}(y) = \bigwedge\{x \in \mathcal{L}_1 : \psi(x) \preceq y\} \; \forall \; y \in \mathcal{L}_2$.
4. $\overline{\psi\cdot}(y) = \bigvee\{x \in \mathcal{L}_1 : \psi(x) \preceq y\} \; \forall \; y \in \mathcal{L}_2$.
5. $\overline{\alpha\beta}(x) = \bigvee\{y \in \mathcal{L}_2 : \alpha(y) \preceq x \preceq \beta(y)\} \; \forall \; x \in \mathcal{L}_1$.
6. $\underline{\alpha\beta}(x) = \bigwedge\{y \in \mathcal{L}_2 : \alpha(y) \preceq x \preceq \beta(y)\} \; \forall \; x \in \mathcal{L}_1$.

Also, let $\leq$ be the partial ordering of $\mathcal{O}_{21} \times \mathcal{O}_{21}$, defined by

$$(\alpha, \beta) \leq (\alpha', \beta') \iff \alpha' \preceq_p \alpha \text{ and } \beta \preceq_p \beta'.$$

Definition 11 *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete lattices, let $\mathcal{MC}_{12}(\mathbf{\Delta}_{21})$ denote the set of morphological connections $(\psi, (\alpha, \beta))$ between $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $(\alpha, \beta) \in \mathbf{\Delta}_{21}$.*

Theorem 1 (Banon-Barrera) *If $(\psi, (\alpha, \beta)) \in \mathcal{MC}_{12}(\mathbf{\Delta}_{21})$, then $\alpha = \underline{\cdot\psi}$, $\beta = \overline{\cdot\psi}$, $\psi = \overline{\alpha\beta}$, $\psi \in \mathbf{\Lambda}_{12}$, and $(\alpha, \beta) \in \mathcal{D}\widetilde{\mathcal{D}}_{21}$. In fact, there is a unique $(\psi, (\alpha, \beta)) \in \mathcal{MC}_{12}(\mathbf{\Delta}_{21})$ for every $\psi \in \mathbf{\Lambda}_{12}$; that is, the set of $(\psi, (\alpha, \beta)) \in \mathcal{MC}_{12}(\mathbf{\Delta}_{21})$ is the graph of the function $\psi \longmapsto (\underline{\cdot\psi}, \overline{\cdot\psi})$ that maps $\mathbf{\Lambda}_{12}$ to $\mathcal{D}\widetilde{\mathcal{D}}_{21}$. Indeed we have more:*

(a) *$\psi \longmapsto (\underline{\cdot\psi}, \overline{\cdot\psi})$ is a bijection of $\mathbf{\Lambda}_{12}$ onto $\mathcal{D}\widetilde{\mathcal{D}}_{21}$, whose inverse is $(\alpha, \beta) \longmapsto \overline{\alpha\beta}$.*

(b) *$\psi \longmapsto (\underline{\psi\cdot}, \overline{\psi\cdot})$ is a bijection of $\mathbf{M}_{12}$ onto $\widetilde{\mathcal{E}}\mathcal{E}_{21}$, whose inverse is $(\alpha, \beta) \longmapsto \underline{\alpha\beta}$.*

(c) *$(\psi, (\alpha, \beta)), (\psi', (\alpha', \beta')) \in \mathcal{MC}_{12}(\mathbf{\Delta}_{21}) \Longrightarrow (\psi \preceq_p \psi' \iff (\alpha, \beta) \leq (\alpha', \beta'))$.*

Thus $(\mathbf{\Lambda}_{12}, \preceq_p)$ and $(\mathbf{M}_{12}, \preceq_p)$ are poset isomorphic to $(\mathcal{D}\widetilde{\mathcal{D}}_{21}, \leq)$ and $(\widetilde{\mathcal{E}}\mathcal{E}_{21}, \leq)$, respectively, and it consequently follows that

(d) *$\mathcal{D}\widetilde{\mathcal{D}}_{21}$ and $\widetilde{\mathcal{E}}\mathcal{E}_{21}$ are complete lattices relative to $\leq$.*

(e) *$\psi \longmapsto (\underline{\cdot\psi}, \overline{\cdot\psi})$ is a lattice isomorphism of $\mathbf{\Lambda}_{12}$ onto $\mathcal{D}\widetilde{\mathcal{D}}_{21}$.*

(f) *$\psi \longmapsto (\underline{\psi\cdot}, \overline{\psi\cdot})$ is a lattice isomorphism of $\mathbf{M}_{12}$ onto $\widetilde{\mathcal{E}}\mathcal{E}_{21}$.*

Corollary 1 *The Achache-Serra propositions are corollaries of this theorem. Specifically,*

1. *The set of Galois connections between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the graph of the lattice isomorphism between $\widetilde{\mathcal{D}}_{12}$ and $\widetilde{\mathcal{D}}_{21}$ given for all $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ and $\widetilde{\varsigma} \in \widetilde{\mathcal{D}}_{21}$ by $\widetilde{\delta} \longmapsto \overline{\cdot\widetilde{\delta}}$ and $\widetilde{\varsigma} \longmapsto \overline{\mathbf{O}\widetilde{\varsigma}}$.*

2. *The set of adjunctions between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the graph of the dual-lattice isomorphism between $\mathcal{E}_{12}$ and $\mathcal{D}_{21}$ given for all $\varepsilon \in \mathcal{E}_{12}$ and $\varsigma \in \mathcal{D}_{21}$ by $\varepsilon \longmapsto \underline{\cdot\varepsilon}$ and $\varsigma \longmapsto \overline{\varsigma\mathbf{E}}$.*

2.7 Banon-Barrera Decompositions

Proposition 12 *If $\psi \in \mathcal{O}_{12}$, then we have the following.*

(a) *ψ is sup-generating if and only if there is an $\varepsilon \in \mathcal{E}_{12}$ and a $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ such that $\psi = \varepsilon \wedge \widetilde{\delta}$.*

(b) *ψ is inf-generating if and only if there is a $\delta \in \mathcal{D}_{12}$ and an $\widetilde{\varepsilon} \in \widetilde{\mathcal{E}}_{12}$ such that $\psi = \delta \vee \widetilde{\varepsilon}$.*

As a consequence of these results we obtain, in particular, the following:

(c) *If $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}_{21}$, then $\overline{\varsigma\widetilde{\varsigma}} \in \mathbf{\Lambda}_{12}$ and can therefore be expressed as the infimum of an erosion and an anti-dilation in $\mathcal{O}_{12}$; such an expression, in fact, is $\overline{\varsigma\widetilde{\varsigma}} = \overline{\varsigma\mathbf{E}} \wedge \overline{\mathbf{O}\widetilde{\varsigma}}$.*

(d) *If* $(\tilde{\epsilon}, \epsilon) \in \tilde{\mathcal{E}}\mathcal{E}_{21}$, *then* $\underline{\tilde{\epsilon}\epsilon} \in \mathbf{M}_{12}$ *and can therefore be expressed as the supremum of a dilation and an anti-erosion in* $\mathcal{O}_{12}$*; such an expression, in fact, is* $\underline{\tilde{\epsilon}\epsilon} = \underline{\tilde{\epsilon}\mathbf{E}} \vee \underline{\mathbf{O}\epsilon}$.

Proposition 13 *The subsets* $\mathbf{\Lambda}_{12}$ *and* $\mathbf{M}_{12}$ *of* $\mathcal{O}_{12}$ *are, respectively, sup-generating and inf-generating. That is,* $\psi = \bigvee\{\lambda \in \mathbf{\Lambda}_{12} : \lambda \preceq_p \psi\} = \bigwedge\{\mu \in \mathbf{M}_{12} : \psi \preceq_p \mu\}$ *for all* $\psi \in \mathcal{O}_{12}$.

Definition 12 *Let* $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *denote the set of functions* $\mathcal{F} : \mathcal{L}_2 \longrightarrow \mathcal{P}(\mathcal{L}_1)$, *and let* $\ll$ *denote the partial ordering of* $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *defined by* $\mathcal{F} \ll \mathcal{F}' \iff \mathcal{F}(y) \subset \mathcal{F}'(y)$ *for all* $y \in \mathcal{L}_2$.

Note that $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ inherits the complete lattice structure of $\mathcal{P}(\mathcal{L}_1)$ (the power set of $\mathcal{L}_1$).

Definition 13 *Let* $\cdot\mathcal{K} : \mathcal{O}_{12} \longrightarrow \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *and* $\mathcal{K}\cdot : \mathcal{O}_{12} \longrightarrow \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *be defined for* $\psi \in \mathcal{O}_{12}$ *and* $y \in \mathcal{L}_2$ *by* $\cdot\mathcal{K}(\psi)(y) = \{x \in \mathcal{L}_1 : y \preceq \psi(x)\}$ *and* $\mathcal{K} \cdot (\psi)(y) = \{x \in \mathcal{L}_1 : \psi(x) \preceq y\}$. *We call* $\cdot\mathcal{K}(\psi)$ *and* $\mathcal{K} \cdot (\psi)$ *the* **left-kernel** *and* **right-kernel**, *respectively, of* ψ.

Definition 14 *Let* $\overline{\kappa}, \underline{\kappa} : \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2} \longrightarrow \mathcal{O}_{12}$ *be defined for* $\mathcal{F} \in \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *and* $x \in \mathcal{L}_1$ *by*

$$\overline{\kappa}(\mathcal{F})(x) = \bigvee\{y \in \mathcal{L}_2 : x \in \mathcal{F}(y)\} \quad \text{and} \quad \underline{\kappa}(\mathcal{F})(x) = \bigwedge\{y \in \mathcal{L}_2 : x \in \mathcal{F}(y)\}.$$

We say that the mappings $\overline{\kappa}(\mathcal{F})$ *and* $\underline{\kappa}(\mathcal{F})$ *are* **derived from** $\mathcal{F}$.

Proposition 14 *For the pairs* $(\cdot\mathcal{K}, \overline{\kappa})$ *and* $(\mathcal{K}\cdot, \underline{\kappa})$ *we have the following.*

1. $\cdot\mathcal{K}$ *maps* $\mathcal{O}_{12}$ *one-to-one into* $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$, *and* $\overline{\kappa}$ *is the inverse of* $\cdot\mathcal{K}$ *on* $\cdot\mathcal{K}(\mathcal{O}_{12})$*; that is,*

$$\overline{\kappa}(\cdot\mathcal{K}(\psi)) = \psi \text{ for all } \psi \in \mathcal{O}_{12}.$$

2. $\mathcal{K}\cdot$ *maps* $\mathcal{O}_{12}$ *one-to-one into* $\mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$, *and* $\underline{\kappa}$ *is the inverse of* $\mathcal{K}\cdot$ *on* $\mathcal{K} \cdot (\mathcal{O}_{12})$*; that is,*

$$\underline{\kappa}(\mathcal{K} \cdot (\psi)) = \psi \text{ for all } \psi \in \mathcal{O}_{12}.$$

3. *If* $\psi, \varphi \in \mathcal{O}_{12}$, *then* $\psi \preceq_p \varphi \iff \cdot\mathcal{K}(\psi) \ll \cdot\mathcal{K}(\varphi) \iff \mathcal{K} \cdot (\psi) \ll \mathcal{K} \cdot (\varphi)$.

Definition 15 *A function* $\mathcal{I}$ *defined on* $\mathcal{L}_2$ *with values in* $\mathcal{P}(\mathcal{L}_1)$ *is called an* **interval function** *if* $\mathcal{I}(y)$ *is either* $\emptyset$ *or a closed interval of* $\mathcal{L}_1$ *for all* $y \in \mathcal{L}_2$.

Definition 16 *For each* $(\alpha, \beta) \in \mathbf{\Delta}_{21}$, *define the interval function* $[\alpha, \beta]$ *for* $y \in \mathcal{L}_2$ *by*

$$[\alpha, \beta](y) = \begin{cases} [\alpha(y), \beta(y)] & \text{if } \alpha(y) \preceq \beta(y) \\ \emptyset & \text{otherwise.} \end{cases}$$

To each interval function $\mathcal{I}$, *associate the unique* $(\alpha_{\mathcal{I}}, \beta_{\mathcal{I}}) \in \mathbf{\Delta}_{21}$ *given for all* $y \in \mathcal{L}_2$ *by* $\alpha_{\mathcal{I}}(y) = \inf \mathcal{I}(y)$ *and* $\beta_{\mathcal{I}}(y) = \sup \mathcal{I}(y)$*;* $\alpha_{\mathcal{I}}$ *and* $\beta_{\mathcal{I}}$ *are called the* **extremities of** $\mathcal{I}$.

Remark 6 *The mappings* $(\alpha,\beta) \longmapsto [\alpha,\beta]$ *and* $\mathcal{I} \longmapsto (\alpha_{\mathcal{I}},\beta_{\mathcal{I}})$, *respectively, on* $\boldsymbol{\Delta}_{21}$ *and the set of interval functions, are reciprocal, i.e., under the first,* $(\alpha_{\mathcal{I}},\beta_{\mathcal{I}}) \longmapsto \mathcal{I}$ (or otherwise stated, $[\alpha_{\mathcal{I}},\beta_{\mathcal{I}}] = \mathcal{I}$), *and under the second,* $[\alpha,\beta] \longmapsto (\alpha,\beta)$.

Proposition 15 *If* $\psi \in \mathcal{O}_{12}$, *then we have the following.*

1. ψ *is sup-generating if and only if its left-kernel* $\cdot\mathcal{K}(\psi)$ *is an interval function.*

2. ψ *is inf-generating if and only if its right-kernel* $\mathcal{K}\cdot(\psi)$ *is an interval function.*

3. $\psi \in \boldsymbol{\Lambda}_{12} \Longrightarrow \cdot\mathcal{K}(\psi) = [\cdot\underline{\psi}, \cdot\overline{\psi}]$ *and* $\psi \in \mathbf{M}_{12} \Longrightarrow \mathcal{K}\cdot(\psi) = [\underline{\psi}\cdot, \overline{\psi}\cdot]$.

Theorem 2 (Decomposition theorem) *If* $\psi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$, *then we have the following.*

(a) $\psi = \bigvee\{\underline{\varsigma\mathbf{E}} \wedge \overline{\mathbf{O}\tilde{\varsigma}} : (\varsigma,\tilde{\varsigma}) \in \mathcal{D}\tilde{\mathcal{D}}_{21}$ *and* $[\varsigma,\tilde{\varsigma}] \ll \cdot\mathcal{K}(\psi)\}$.

(b) $\psi = \bigwedge\{\underline{\tilde{\epsilon}\mathbf{E}} \vee \underline{\mathbf{O}\epsilon} : (\tilde{\epsilon},\epsilon) \in \tilde{\mathcal{E}}\mathcal{E}_{21}$ *and* $[\tilde{\epsilon},\epsilon] \ll \mathcal{K}\cdot(\psi)\}$.

Specialized versions of the above theorem can be developed for the inf-separable, sup-separable, increasing, and decreasing mappings with the aid of the following concepts.

Definition 17 *If* $\gamma \in \mathcal{O}_{21}$, $\mathcal{F} \in \mathcal{P}(\mathcal{L}_2)^{\mathcal{L}_1}$, *and* $\emptyset \notin \mathcal{F}(\mathcal{L}_2)$, *then the expression* $\gamma(\in)\mathcal{F}$ *is used as shorthand for* $\gamma(y) \in \mathcal{F}(y)$ *for all* $y \in \mathcal{L}_2$. *In addition, it is said that*

1. $\mathcal{F}$ *is* **convex** *if* $\alpha,\beta(\in)\mathcal{F}$ *and* $\alpha \preceq \beta \Longrightarrow [\alpha,\beta] \ll \mathcal{F}$.

2. $\mathcal{F}$ *is* $\vee$**-hereditary** *if* $\alpha(\in)\mathcal{F}$ *and* $\alpha \preceq \beta \Longrightarrow \beta(\in)\mathcal{F}$.

Remark 7 $\mathcal{F}$ *is* $\vee$*-hereditary* $\Longrightarrow$ $\mathcal{F}$ *is convex.*

Proposition 16 *If* $\psi \in \mathcal{O}_{12}$ *is inf-separable* (sup-separable), *and if there is an* $x \in \mathcal{L}_1$ *such that* $\psi(x) = e$ $(\psi(x) = o)$, *then* $\cdot\mathcal{K}(\psi)$ $(\mathcal{K}\cdot(\psi))$ *is convex. In addition, we have the following.*

1. *If* $\psi \in \mathcal{O}_{12}^{+}$ *and* $\psi(E) = e$, *then* $\cdot\mathcal{K}(\psi)$ *is* $\vee$*-hereditary.*

2. *If* $\psi \in \mathcal{O}_{12}^{-}$ *and* $\psi(O) = e$, *then* $\mathcal{K}\cdot(\psi)$ *is* $\vee$*-hereditary.*

The following is now a corollary of Theorem 2.

Corollary 2 (Special decompositions) *Let* $\psi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$.

(a) *If* ψ *is inf-separable and there is an* $x \in \mathcal{L}_1$ *such that* $\psi(x) = e$, *then*

$$\psi = \bigvee\{\underline{\varsigma\mathbf{E}} \wedge \overline{\mathbf{O}\tilde{\varsigma}} : (\varsigma,\tilde{\varsigma}) \in \mathcal{D}\tilde{\mathcal{D}}_{21} \text{ and } \varsigma,\tilde{\varsigma}(\in)\cdot\mathcal{K}(\psi)\}.$$

(b) *If ψ is increasing and such that $\psi(E) = e$, then*

$$\psi = \bigvee\{\overline{\varsigma\mathbf{E}} : \varsigma \in \mathcal{D}_{21} \text{ and } \varsigma(\in) \cdot \mathcal{K}(\psi)\} \quad \textit{and} \quad \psi = \bigwedge\{\underline{\mathbf{O}\epsilon} : \epsilon \in \mathcal{E}_{21} \text{ and } \epsilon(\in)\mathcal{K} \cdot (\psi)\}.$$

(c) *If ψ is sup-separable and there is an $x \in \mathcal{L}_1$ such that $\psi(x) = o$, then*

$$\psi = \bigwedge\{\underline{\widetilde{\epsilon}\mathbf{E}} \vee \underline{\mathbf{O}\epsilon} : (\widetilde{\epsilon}, \epsilon) \in \widetilde{\mathcal{E}}\mathcal{E}_{21} \text{ and } \widetilde{\epsilon}, \epsilon(\in)\mathcal{K} \cdot (\psi)\}.$$

(d) *If ψ is decreasing and such that $\psi(O) = e$, then*

$$\psi = \bigvee\{\overline{\mathbf{O}\widetilde{\varsigma}} : \widetilde{\varsigma} \in \widetilde{\mathcal{D}}_{21} \text{ and } \widetilde{\varsigma}(\in) \cdot \mathcal{K}(\psi)\} \quad \textit{and} \quad \psi = \bigwedge\{\underline{\widetilde{\epsilon}\mathbf{E}} : \widetilde{\epsilon} \in \widetilde{\mathcal{E}}_{21} \text{ and } \widetilde{\epsilon}(\in)\mathcal{K} \cdot (\psi)\}.$$

With apologies for its density, this completes my synopsis of the Banon-Barrera theory.

3 Heijmanns-Ronse Lattice Morphology

I now take up the group theoretical approach of Heijmanns and Ronse to complete-lattice morphology. For this it is well to start with the general concept of a **group action** on a set.

Remark 8 *Let X be a nonempty set and let $\mathcal{T}(X)$ denote the set of transformations of X, i.e., the set of bijections of X onto X. Then $(\mathcal{T}(X), \circ)$ is a group called the* **transformation group** *of X. A subgroup of $(\mathcal{T}(X), \circ)$ is called a* **group of transformations** *of X.*

Definition 18 *Let X be a nonempty set, let $(G, \cdot)$ be a group, let $\mathcal{H} : g \longmapsto \sigma_g$ be a homomorphism of G onto a group $(\{\sigma_g : g \in G\}, \circ)$ of transformations of X, and define the mapping $\sigma : G \times X \longrightarrow X$ by $\sigma(g, x) = \sigma_g(x)$. Then (G, σ) is called* **a group action on** X.

Definition 19 *Let (G, σ) be a group action on X.*

1. *If $\sigma_g(x) = x$ for all $x \in X$ implies that g is the identity element of G, then (G, σ) is called an* **effective group action** *on X.*

2. *If for all $x, x' \in X$, there exists a $g \in G$ such that $\sigma_g(x) = x'$, then (G, σ) is called a* **transitive group action** *on X.*

Remark 9 *Let (G, σ) be a group action on X.*

1. *$\sigma_{g \cdot h} = \sigma_g \circ \sigma_h$ for all $g, h \in G$.*

2. *If ϵ is the identity element of G, then σ_ϵ is the identity element of $(\{\sigma_g : g \in G\}, \circ)$, i.e., $\sigma_\epsilon \in \mathcal{T}(X)$ maps X onto itself identically.*

3. *$\sigma_{g^{-1}} = \sigma_g^{-1}$ for all $g \in G$.*

4. *The homomorphism $\mathcal{H} : g \longmapsto \sigma_g$ is an isomorphism if and only if (G, σ) is an effective group action on X.*

Remark 10 *If $(\{\sigma_\alpha : \alpha \in A\}, \circ)$ is a group of transformations of a set X, and if $(G, \cdot)$ is a group isomorphic as such to $(\{\sigma_\alpha : \alpha \in A\}, \circ)$, then there is an effective action (G, σ) on X such that $(\{\sigma_g : g \in G\}, \circ)$ is group isomorphic to $(\{\sigma_\alpha : \alpha \in A\}, \circ)$.*

If X is a set with a mathematical structure (relational, algebraic, topological, etc), then the invariance properties of that structure are essentially specified by the group $\mathrm{Aut}(X)$ of automorphisms of X, that is, for instance, by the group of poset automorphisms if X is a poset, lattice automorphisms if X is a lattice, homemorphisms if X is a topological space. Indeed, the invariance properties of X's structure can be studied in detail by analyzing $\mathrm{Aut}(X)$ into its subgroups, and herein lies the main utility of group actions, for each subgroup G of $\mathrm{Aut}(X)$ acts effectively on X via the mapping $(g, x) \longmapsto g(x)$. Heijmanns and Ronse begin by applying the automorphism subgroup/group-action concept to complete lattices.

Definition 20 *Let* $\mathrm{Aut}(\mathcal{L})$ *denote the automorphisms of a lattice* $\mathcal{L}$. *If* (G, σ) *is an effective group action on* $\mathcal{L}$, *and if* $\sigma_g \in \mathrm{Aut}(\mathcal{L})$ *for all* $g \in G$, *then* (G, σ) *is said* **to act effectively as a group of automorphisms on** $\mathcal{L}$.

Remark 11 *If* G *is a group,* $\mathcal{L}$ *is a lattice, and* $\sigma : G \times \mathcal{L} \longrightarrow \mathcal{L}$, *then* (G, σ) *acts effectively as a group of automorphisms on* $\mathcal{L}$ *if and only if the following hold.*

1. $g \longmapsto \sigma_g$ *is an isomorphism of* G.

2. σ_g *is an automorphism of* $\mathcal{L}$ *for each* $g \in G$.

Since the automorphisms of a lattice $\mathcal{L}$ form a group under composition, $\{\sigma_g : g \in G\}$ is a subgroup of $\mathrm{Aut}(\mathcal{L})$ whenever (G, σ) acts effectively as a group of automorphisms on $\mathcal{L}$. On the other hand, every subgroup G of $\mathrm{Aut}(\mathcal{L})$ acts effectively as a group of automorphisms on $\mathcal{L}$ via the mapping $\sigma : (g, x) \longmapsto g(x)$, as already noted. Thus the two concepts "subgroup of $\mathrm{Aut}(\mathcal{L})$" and "effective action as a group of automorphisms" are essentially the same.

An example of this concept is provided by *Euclidean morphology*, whose basic object is the complete lattice $\mathcal{P}(\Re^2)$ of subsets of the Euclidean plane $(\Re^2)$, where set intersection and union are the meet and join operations. It is indeed readily seen that the group of planar translations, which plays an essential role in Euclidean morphology, acts effectively as a group of automorphisms on $\mathcal{P}(\Re^2)$. The innovation of Heijmanns and Ronse arose from their discovery of how this type of action could be abstracted to the general complete lattice $\mathcal{L}$. They accomplished this by assuming that $\mathcal{L}$ is not only acted upon effectively by an **abelian** group, G, of automorphisms, but that $\mathcal{L}$ also has a sup-generating subset ℓ, and that ℓ and G satisfy certain compatibility conditions, which are as follows.

Definition 21 *If* $\mathcal{L}$ *is a complete lattice with a sup-generating subset* ℓ, *if* (G, σ) *acts effectively as a group of automorphisms on* $\mathcal{L}$, *and if* G **is an abelian group**, *then we say that the action* (G, σ) *is* ℓ**-admissable** *if the following hold.*

1. ℓ *is* G*-invariant, i.e.,* $\sigma_g(\xi) \in \ell$ *for all* $g \in G$ *and* $\xi \in \ell$.

2. (G, σ) *acts transitively on* ℓ, *i.e., if* $(\xi, \eta) \in \ell \times \ell$, *then* $\sigma_g(\xi) = \eta$ *for some* $g \in G$.

Proposition 17 *If* $\mathcal{L}$ *is a complete lattice with a sup-generating subset* ℓ, *and if* (G, σ) *acts* ℓ*-admissably on* $\mathcal{L}$, *then we have the following.*

1. o *and* e *are not in* ℓ.

2. *Every atom of* $\mathcal{L}$ *lies in* ℓ.

Proof (1) $\sigma_g(o) = o$ and $\sigma_g(e) = e$ for all $g \in G$ because the σ_g are lattice automorphisms. Hence $o, e \notin \ell$ by (2) of the above definition. (2) A stronger result is, in fact, available from general lattice theory; namely: *If* L *is a lattice with a universal lower bound* o *and a sup-generating subset* $\mathcal{X}$, *then every atom of* L *lies in* $\mathcal{X}$. The proof of this is as follows. If $\hat{x}$ is an atom of L, then $\hat{x} = \sup\{\eta \in \mathcal{X} : \eta \preceq \hat{x}\}$. Since $\hat{x}$ is an atom, each $\eta \in \mathcal{X}$ such that $\eta \preceq \hat{x}$ must be either o or $\hat{x}$. Therefore, if $\hat{x} \notin \mathcal{X}$, then $\hat{x}$ is either $\sup\{o\} = o$ or $\sup \emptyset = o$. Since $\hat{x} \neq o$, this is absurd. Hence $\hat{x} \in \mathcal{X}$.

Proposition 18 (Heijmans-Ronse) *If $\mathcal{L}$ is a complete lattice with a sup-generating subset ℓ, and if (G,σ) acts ℓ-admissably on $\mathcal{L}$, then (G,σ)* **acts regularly on** *ℓ, i.e., if $(\xi,\eta) \in \ell \times \ell$, then $\sigma_g(\xi) = \eta$ for a unique $g = g_{\xi\eta} \in G$. In other words, there is a function $(\xi,\eta) \longmapsto \sigma_{g_{\xi\eta}}$ on $\ell \times \ell$ with values in $\{\sigma_g : g \in G\}$ (equivalently, with values in G) such that $\sigma_{g_{\xi\eta}}(\xi) = \eta$.*

Corollary 3 *The function defined on $\ell \times \ell$ by $(\xi,\eta) \longmapsto g_{\xi\eta}$ is onto G; in fact, for each fixed $\xi \in \ell$ the function $\eta \longmapsto g_{\xi\eta}$ is a bijection of ℓ onto G.*

Proof It is clearly sufficient to prove the second assertion. Given $g \in G$ and $\xi \in \ell$, we have that $\sigma_g(\xi) \in \ell$ by the G-invariance of ℓ. Let η be the element of ℓ equal to $\sigma_g(\xi)$. Then it follows from the uniqueness assertion of the proposition that $g = g_{\xi\eta}$. Hence g is in the range of $\eta \longmapsto g_{\xi\eta}$ for all $\xi \in \ell$. Now suppose that $g_{\xi\eta} = g_{\xi\zeta}$. This implies that $\sigma_{g_{\xi\eta}} = \sigma_{g_{\xi\zeta}}$, which in turn implies that $\eta = \sigma_{g_{\xi\eta}}(\xi) = \sigma_{g_{\xi\zeta}}(\xi) = \zeta$. Hence $\eta \longmapsto g_{\xi\eta}$ is a bijection of ℓ onto G for each fixed $\xi \in \ell$.

Definition 22 *Let $\mathcal{L}$ be a complete lattice with a sup-generating subset ℓ, and let (G,σ) act ℓ-admissably on $\mathcal{L}$. With respect to a* **fixed reference element** *$\rho \in \ell$,* **define the operators** *$\tau_\eta = \sigma_{g_{\rho\eta}}$ for $\eta \in \ell$ and denote the set $\{\tau_\eta : \eta \in \ell\}$ by $\mathbf{T}$.*

Remark 12 *$\eta \longmapsto \tau_\eta$ is a bijection of ℓ onto $\mathbf{T}$ and $(\mathbf{T},\circ) = (\{\sigma_g : g \in G\},\circ)$.*

Definition 23 *Let $\mathcal{L}$ be a complete lattice with a sup-generating subset ℓ, let (G,σ) act ℓ-admissably on $\mathcal{L}$, and let $\rho \in \ell$ be a fixed reference element. With respect to ρ,* **define an addition operation $+$ in** *ℓ by setting $\xi + \eta = \tau_\xi\tau_\eta(\rho) = \tau_\xi(\eta)$ for each $(\xi,\eta) \in \ell \times \ell$.*

Since $\tau_\xi\tau_\eta(\rho) = \tau_\eta\tau_\xi(\rho) = \tau_\eta(\xi) = \eta + \xi$, we see that the $+$ operation is commutative. Indeed, **$(\ell,+)$ is a commutative group with ρ the additive identity and $-\xi \equiv \tau_\xi^{-1}(\rho)$ the additive inverse of** ξ. In fact we have the following.

Proposition 19 *The bijection $\eta \longmapsto \tau_\eta$ of ℓ onto $\mathbf{T}$ is a group isomorphism of $(\ell,+)$ onto $(\mathbf{T},\circ)$ and a poset isomorphism of $(\ell,\preceq)$ onto $(\mathbf{T},\preceq_p)$.*

Proof If $\xi,\eta \in \ell$, then $\xi + \eta \longmapsto \tau_{\xi+\eta}$. Since $\tau_{\xi+\eta}(\rho) = \xi + \eta = \tau_\xi\tau_\eta(\rho)$, it follows that $\tau_{\xi+\eta}$ and $\tau_\xi\tau_\eta$ are the same automorphism. Hence $\eta \longmapsto \tau_\eta$ is a group isomorphism.

To complete the proof we show that if $\xi,\eta \in \ell$, $\xi \preceq \eta$, and $x \in \mathcal{L}$, then $\tau_\xi(x) \preceq \tau_\eta(x)$. For this, first note that $\xi \preceq \eta \Longrightarrow \lambda + \xi = \tau_\lambda(\xi) \preceq \tau_\lambda(\eta) = \lambda + \eta$ for all $\lambda \in \ell$. Since $+$ is commutative, we therefore have $\xi + \lambda = \tau_\xi(\lambda) \preceq \tau_\eta(\lambda) = \eta + \lambda$, i.e., the proposition holds if $x \in \ell$. If $x \in \mathcal{L}$, we have $\tau_\xi(x) = \tau_\xi(\bigvee\{\mu \in \ell : \mu \preceq x\}) =$

$$\bigvee\{\tau_\xi(\mu) : \mu \in \ell, \mu \preceq x\} = \bigvee\{\xi + \mu : \mu \in \ell, \mu \preceq x\} \preceq$$

$$\preceq \bigvee\{\eta + \mu : \mu \in \ell, \mu \preceq x\} = \tau_\eta(x).$$

3.1 G-Invariant Lattice Mappings

The next step in the Heijmanns-Ronse approach is to study the class of G-invariant lattice mappings. Consider first the invariant operators in $\mathcal{O} = \mathcal{O}(\mathcal{L})$.

Definition 24 *If $\mathcal{L}$ is a complete lattice and (G, σ) acts as a group of automorphisms on $\mathcal{L}$, then a $\psi \in \mathcal{O}(\mathcal{L})$ is said to be G-**invariant** if $\psi(\sigma_g(x)) = \sigma_g(\psi(x))$ for all $x \in \mathcal{L}$ and $g \in G$.*

To define invariant mappings in $\mathcal{O}_{12}$ or $\mathcal{O}_{21}$, **assume that** G is a group that **acts** effectively as a group of automorphisms **on both** $\mathcal{L}_1$ **and** $\mathcal{L}_2$; **that is**, assume **there are mappings**

$$s : G \times \mathcal{L}_1 \longrightarrow \mathcal{L}_1 \quad \text{and} \quad \sigma : G \times \mathcal{L}_2 \longrightarrow \mathcal{L}_2$$

such that (1) $g \longmapsto s_g$ and $g \longmapsto \sigma_g$ are isomorphisms of G, (2) s_g is an automorphism of $\mathcal{L}_1$ for all $g \in G$, and (3) σ_g is an automorphism of $\mathcal{L}_2$ for all $g \in G$.

Definition 25 *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete lattices, and let G be a group that acts effectively as a group of automorphisms on both $\mathcal{L}_1$ and $\mathcal{L}_2$. Then we adopt the following definitions.*

1. *A $\psi \in \mathcal{O}_{12}$ is said to be G-**invariant** if $\psi(s_g(x)) = \sigma_g(\psi(x))$ for all $x \in \mathcal{L}_1$ and $g \in G$.*
2. *A $\psi \in \mathcal{O}_{21}$ is said to be G-**invariant** if $\psi(\sigma_g(y)) = s_g(\psi(y))$ for all $y \in \mathcal{L}_2$ and $g \in G$.*

The above definitions become quite fruitful when it is further assumed that $\mathcal{L}_2$ has a sup-generating subset ℓ such that (G, σ) acts ℓ-admissably on $\mathcal{L}_2$; in the case of operators on $\mathcal{L}$, the like assumption is: $\ell \subset \mathcal{L}$ is sup-generating and (G, σ) acts ℓ-admissably on $\mathcal{L}$. (Keep in mind that ℓ-admissable actions (G, σ) require the commutativity of G.) Unless otherwise stated, then, **we will assume the following for the remainder of this report**.

1. $\mathcal{L}_1$ and $\mathcal{L}_2$ are complete lattices.
2. G is an abelian group that acts effectively as a group of automorphisms on both $\mathcal{L}_1$ and $\mathcal{L}_2$ via the maps s and σ.
3. $\mathcal{L}_2$ has a sup-generating subset ℓ such that (G, σ) acts ℓ-admissably on $\mathcal{L}_2$.
4. ρ is a fixed reference element of ℓ.

Besides the operators $\tau_\eta = \sigma_{g_{\rho\eta}}$, $\eta \in \ell$, **we additionally define** $\theta_\eta = s_{g_{\rho\eta}}$ for each $\eta \in \ell$ and denote the set $\{\theta_\eta : \eta \in \ell\}$ by $\mathbf{\Theta}$. While

$$(\mathbf{T}, \circ) = (\{\tau_\eta : \eta \in \ell\}, \circ) = (\{\sigma_g : g \in G\}, \circ) \simeq (G, \cdot)$$

is just another way to designate the pertinent automorphism group of $\mathcal{L}_2$ (where $\simeq$ means "is group isomorphic to"), we likewise have that

$$(\mathbf{\Theta}, \circ) = (\{\theta_\eta : \eta \in \ell\}, \circ) = (\{s_g : g \in G\}, \circ) \simeq (G, \cdot)$$

is just another way to designate the corresponding isomorphic automorphism group of $\mathcal{L}_1$.

Proposition 20 *Indeed, we have the following:*

1. $(\boldsymbol{\Theta}, \circ)$ *is an abelian group of automorphisms of* $\mathcal{L}_1$.
2. $\tau_\eta \longmapsto \theta_\eta$ *is a group isomorphism of* $(\mathbf{T}, \circ)$ *onto* $(\boldsymbol{\Theta}, \circ)$.
3. $\theta_\eta \longmapsto s_{g_{\rho\eta}}$ *is a group isomorphism of* $(\boldsymbol{\Theta}, \circ)$ *onto* $(\{s_g : g \in G\}, \circ)$.
4. $\eta \longmapsto \theta_\eta$ *is a group isomorphism of* $(\ell, +)$ *onto* $(\boldsymbol{\Theta}, \circ)$.

Proof (1) Since by definition $\theta_\eta = s_{g_{\rho\eta}}$ for all $\eta \in \ell$, it is clear that $\boldsymbol{\Theta}$ is a set of automorphisms of $\mathcal{L}_1$. The rest of (1) is established in (2) below. (2) $\tau_\eta \longmapsto \theta_\eta$ is clearly onto $\boldsymbol{\Theta}$. If $\theta_\eta = \theta_\xi$, then $s_{g_{\rho\eta}} = s_{g_{\rho\xi}}$, and it follows that $g_{\rho\eta} = g_{\rho\xi}$ because $g \longmapsto s_g$ is a group isomorphism. Since $\eta \longmapsto g_{\rho\eta}$ is a bijection of ℓ onto G, it follows that $\eta = \xi$, and hence that $\tau_\eta \longmapsto \theta_\eta$ is a bijection of $\mathbf{T}$ onto $\boldsymbol{\Theta}$. If $\eta, \xi \in \ell$, then $\tau_\eta \circ \tau_\xi = \tau_{\eta+\xi} \longmapsto \theta_{\eta+\xi} = s_{g_{\rho(\eta+\xi)}} = s_{g_{\rho\eta} \cdot g_{\rho\xi}} = s_{g_{\rho\eta}} \circ s_{g_{\rho\xi}} = \theta_\eta \circ \theta_\xi$. (3) $\theta_\eta \longmapsto s_{g_{\rho\eta}}$ is clearly a bijection of $\boldsymbol{\Theta}$ onto $\{s_g : g \in G\}$; moreover, if $\eta, \xi \in \ell$, then $\theta_\eta \circ \theta_\xi = s_{g_{\rho\eta}} \circ s_{g_{\rho\xi}}$ by definition. (4) Since $\eta \longmapsto \tau_\eta$ is a group isomorphism of $(\ell, +)$ onto $(\mathbf{T}, \circ)$, and since $\tau_\eta \longmapsto \theta_\eta$ is a group isomorphism of $(\mathbf{T}, \circ)$ onto $(\boldsymbol{\Theta}, \circ)$, it is clear that $\eta \longmapsto \theta_\eta$, being a composition of group isomorphisms, is a group isomorphism of $(\ell, +)$ onto $(\boldsymbol{\Theta}, \circ)$.

Corollary 4 *Thus we obtain the following equivalent characterizations of G-invariance.*

1. $\psi \in \mathcal{O}_{12}$ **is** G**-invariant if and only if** $\psi(\theta_\eta(x)) = \tau_\eta(\psi(x))$ *for all* $x \in \mathcal{L}_1$ *and* $\eta \in \ell$.
2. $\psi \in \mathcal{O}_{21}$ **is** G**-invariant if and only if** $\psi(\tau_\eta(y)) = \theta_\eta(\psi(y))$ *for all* $y \in \mathcal{L}_2$ *and* $\eta \in \ell$.

3.2 G-Invariance and the Banon-Barrera Theory

Theorem 3 *Adjunctions, Galois connections, and morphological connections between* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *have the following G-invariance properties.*

1. *If* (ε, ς) *is an adjunction between* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, then the following are equivalent.*
   (a) ε *is G-invariant.*
   (b) ς *is G-invariant.*
2. *If* $(\widetilde{\delta}, \widetilde{\varsigma})$ *is a Galois connection between* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, then the following are equivalent.*
   (c) $\widetilde{\delta}$ *is G-invariant.*
   (d) $\widetilde{\varsigma}$ *is G-invariant.*
3. *If* $(\psi, (\alpha, \beta)) \in \mathcal{MC}_{12}(\boldsymbol{\Delta}_{21})$*, then the following are equivalent.*
   (e) ψ *is G-invariant.*
   (f) α *and* β *are G-invariant.*

Proof (1) Assume first that $\varepsilon \in \mathcal{E}_{12}$ is G-invariant. For all $y \in \mathcal{L}_2$ and all $g \in G$ we have

$$s_g(\varsigma(y)) = s_g \bigwedge\{x \in \mathcal{L}_1 : y \preceq \varepsilon(x)\} = \bigwedge\{s_g(x) : x \in \mathcal{L}_1, y \preceq \varepsilon(x)\},$$

since automorphisms commute with $\bigwedge$. Put $s_g(x) = x'$. Then $x = s_{g^{-1}}(x')$ so that

$$\bigwedge\{s_g(x) : x \in \mathcal{L}_1, y \preceq \varepsilon(x)\} = \bigwedge\{x' \in \mathcal{L}_1 : y \preceq \varepsilon(s_{g^{-1}}(x')\} =$$

$$\bigwedge\{x' \in \mathcal{L}_1 : y \preceq \sigma_{g^{-1}}(\varepsilon(x'))\} = \bigwedge\{x' \in \mathcal{L}_1 : \sigma_g(y) \preceq \varepsilon(x')\} =$$

$$\underline{\varepsilon}(\sigma_g(y)) = \varsigma(\sigma_g(y)).$$

Hence $\varsigma \in \mathcal{D}_{21}$ is G-invariant. The rest of the proof of (1) should now be clear.

(2) Assume first that $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ is G-invariant. For all $y \in \mathcal{L}_2$ and all $g \in G$ we have $s_g(\widetilde{\varsigma}(y)) = \bigvee\{s_g(x) : x \in \mathcal{L}_1, y \preceq \widetilde{\delta}(x)\}$, because automorphisms commute with $\bigvee$. The proof that (c) $\Longrightarrow$ (d) proceeds as above, and the proof of the converse is similar.

(3) Assume first that α and β are G-invariant. For all $x \in \mathcal{L}_1$ and $g \in G$ we have

$$\sigma_g(\psi(x)) = \sigma_g\left(\overline{\alpha\beta}(x)\right) = \bigvee\{\sigma_g(y) : y \in \mathcal{L}_2, \alpha(y) \preceq x \preceq \beta(y)\}.$$

Put $\sigma_g(y) = y'$. Then $y = \sigma_{g^{-1}}(y')$ so that

$$\sigma_g(\psi(x)) = \bigvee\{y' \in \mathcal{L}_2 : \alpha(\sigma_{g^{-1}}(y')) \preceq x \preceq \beta(\sigma_{g^{-1}}(y'))\}$$

$$= \bigvee\{y' \in \mathcal{L}_2 : s_{g^{-1}}(\alpha(y')) \preceq x \preceq s_{g^{-1}}(\beta((y'))\} = \bigvee\{y' \in \mathcal{L}_2 : \alpha(y') \preceq s_g(x) \preceq \beta((y')\}$$

$$= \overline{\alpha\beta}(s_g(x)) = \psi(s_g(x)).$$

Thus $\psi \in \mathbf{\Lambda}_{12}$ is G-invariant. Now assume that $\psi \in \mathbf{\Lambda}_{12}$ is G-invariant. For all $y \in \mathcal{L}_2$ and $g \in G$ we have $s_g(\alpha(y)) = \bigwedge\{s_g(x) : x \in \mathcal{L}_1, y \preceq \psi(x)\}$ and

$$s_g(\beta(y)) = \bigvee\{s_g(x) : x \in \mathcal{L}_1, y \preceq \psi(x)\}.$$

From these it is readily verified as above that $\alpha \in \mathcal{D}_{21}$ and $\beta \in \widetilde{\mathcal{D}}_{21}$ are G-invariant.

Definition 26 *Let us adopt the following terminology.*

1. *If ε and ς are G-invariant and (ε, ς) is an adjunction between $\mathcal{L}_1$ and $\mathcal{L}_2$, then (ε, ς) will be called a* G-**adjunction**.

2. *If $\widetilde{\delta}$ and $\widetilde{\varsigma}$ are G-invariant and $(\widetilde{\delta}, \widetilde{\varsigma})$ is a Galois connection between $\mathcal{L}_1$ and $\mathcal{L}_2$, then $(\widetilde{\delta}, \widetilde{\varsigma})$ will be called a* G-**invariant Galois connection**.

3. *If α, β, and ψ are G-invariant and $(\psi, (\alpha, \beta)) \in \mathcal{MC}_{12}(\mathbf{\Delta}_{21})$, then $(\psi, (\alpha, \beta))$ will be called a* G-**invariant morphological connection**.

4. *If α and β are G-invariant, then we will call* **the ordered pair** (α, β) G-**invariant**.

Let $\mathcal{E}_{12}^G$, $\mathcal{D}_{12}^G$, $\tilde{\mathcal{E}}_{12}^G$, and $\tilde{\mathcal{D}}_{12}^G$ denote, respectively, **the sets of G-invariant erosions, dilations, anti-erosions, and anti-dilations in $\mathcal{O}_{12}$**. In the same way, let $\mathcal{O}_{12}^G$, $\mathcal{O}_{12}^{G+}$, and $\mathcal{O}_{12}^{G-}$, respectively, denote **the G-invariant mappings, the G-invariant increasing mappings, and the G-invariant decreasing mappings in $\mathcal{O}_{12}$**. Our next task is to establish the sublattice properties of these sets, and for this the following lemma will prove useful.

Lemma 1 *Let L, M, and N be complete lattices, let N be a sublattice of M, and let M be a sublattice of L. If N is, additionally, a complete (meet-complete, join-complete) sublattice of L, then N is a complete (meet-complete, join-complete) sublattice of M.*

Proof From general lattice theory, it follows for all $B \subset N$ that

$$\inf_N B \preceq \inf_M B \preceq \inf_L B \preceq \sup_L B \preceq \sup_M B \preceq \sup_N B.$$

Our three alternative hypotheses for N are equivalent to the following:

A. (Complete) $\inf_L B = \inf_N B$ and $\sup_L B = \sup_N B$ for all $B \subset N$.

B. (Meet-Complete) $\inf_L B = \inf_N B$ for all $B \subset N$.

C. (Join-Complete) $\sup_L B = \sup_N B$ for all $B \subset N$.

Note that the truth of the lemma for hypotheses (B) and (C) implies its truth for hypothesis (A). For hypothesis (B), the lemma follows from

$$\inf_N B \preceq \inf_M B \preceq \inf_L B = \inf_N B.$$

For hypothesis (C), the lemma follows from

$$\sup_N B = \sup_L B \preceq \sup_M B \preceq \sup_N B.$$

Proposition 21 *With regard to $\mathcal{O}_{12}^G$, $\mathcal{O}_{12}^{G+}$, and $\mathcal{O}_{12}^{G-}$ we have the following.*

1. *$\mathcal{O}_{12}^G$, $\mathcal{O}_{12}^{G+}$, and $\mathcal{O}_{12}^{G-}$ each contain* $\mathbf{e}$ *and* $\mathbf{o}$.

2. *$\mathcal{O}_{12}^G$, $\mathcal{O}_{12}^{G+}$, and $\mathcal{O}_{12}^{G-}$ are each complete sublattices of $\mathcal{O}_{12}$.*

Proof For all $\eta \in \ell$ and $x \in \mathcal{L}_1$ we have $\tau_\eta(\mathbf{e}(x)) = \tau_\eta(e) = e$, $\mathbf{e}(\theta_\eta(x)) = e$, $\tau_\eta(\mathbf{o}(x)) = \tau_\eta(o) = o$, and $\mathbf{o}(\theta_\eta(x)) = o$. Thus $\mathbf{e}, \mathbf{o} \in \mathcal{O}_{12}^G$; also, it is clear that $\mathbf{e}$ and $\mathbf{o}$ are each both increasing and decreasing.

If $\psi, \varphi \in \mathcal{O}_{12}^G$, then for all $\eta \in \ell$ and $x \in \mathcal{L}_1$ we have $\tau_\eta((\psi \wedge \varphi)(x)) = \tau_\eta(\psi(x) \wedge \varphi(x)) =$

$$\tau_\eta(\psi(x)) \wedge \tau_\eta(\varphi(x)) = \psi(\theta_\eta(x)) \wedge \varphi(\theta_\eta(x)) = (\psi \wedge \varphi)(\theta_\eta(x))$$

and $\tau_\eta((\psi \vee \varphi)(x)) = \tau_\eta(\psi(x) \vee \varphi(x)) =$

$$\tau_\eta(\psi(x)) \vee \tau_\eta(\varphi(x)) = \psi(\theta_\eta(x)) \vee \varphi(\theta_\eta(x)) = (\psi \vee \varphi)(\theta_\eta(x)).$$

Thus $\psi \wedge \varphi$ and $\psi \vee \varphi$ are G-invariant and $\mathcal{O}_{12}^G$ is a sublattice of $\mathcal{O}_{12}$. If $\mathcal{S}$ is a subset of $\mathcal{O}_{12}^G$, then for all $\eta \in \ell$ and $x \in \mathcal{L}_1$ we have:

$$\tau_\eta((\sup_{\mathcal{O}_{12}} \mathcal{S})(x)) = \tau_\eta(\sup_{\mathcal{L}_2}\{\psi(x) : \psi \in \mathcal{S}\}) = \sup_{\mathcal{L}_2}\{\tau_\eta(\psi(x)) : \psi \in \mathcal{S}\}$$

$$= \sup_{\mathcal{L}_2}\{\psi(\theta_\eta(x)) : \psi \in \mathcal{S}\} = (\sup_{\mathcal{O}_{12}}\{\psi \circ \theta_\eta : \psi \in \mathcal{S}\})(x).$$

That is, the supremum of $\mathcal{S}$ in $\mathcal{O}_{12}$ is G-invariant. Thus $\mathcal{O}_{12}^G$ is a join-complete sublattice of $\mathcal{O}_{12}$. The rest of the proof should now be clear.

Corollary 5 *Lemma 1, Proposition 1, and Proposition 21 now give the following results.*

1. *$\mathcal{O}_{12}^{G+}$ is a complete sublattice of $\mathcal{O}_{12}^+$.*

2. *$\mathcal{O}_{12}^{G-}$ is a complete sublattice of $\mathcal{O}_{12}^-$.*

3. *The infimum in $\mathcal{O}_{12}$ of any collection of (decreasing, increasing) G-invariant mappings is a (decreasing, increasing) G-invariant mapping.*

4. *The supremum in $\mathcal{O}_{12}$ of any collection of (decreasing, increasing) G-invariant mappings is a (decreasing, increasing) G-invariant mapping.*

Proposition 22 *For $\mathcal{E}_{12}^G$, $\mathcal{D}_{12}^G$, $\widetilde{\mathcal{E}}_{12}^G$, and $\widetilde{\mathcal{D}}_{12}^G$ we have that*

1. *$\mathcal{E}_{12}^G$ and $\widetilde{\mathcal{D}}_{12}^G$ are meet-complete sublattices of $\mathcal{O}_{12}$.*

2. *$\mathcal{D}_{12}^G$ and $\widetilde{\mathcal{E}}_{12}^G$ are join-complete sublattices of $\mathcal{O}_{12}$.*

Proof We note that $\mathcal{O}_{12}^G$ is a complete sublattice of $\mathcal{O}_{12}$ and that $\mathcal{E}_{12}$ and $\widetilde{\mathcal{D}}_{12}$ are meet-complete sublattices of $\mathcal{O}_{12}$. Since $\mathcal{E}_{12}^G = \mathcal{E}_{12} \cap \mathcal{O}_{12}^G$ and $\widetilde{\mathcal{D}}_{12}^G = \widetilde{\mathcal{D}}_{12} \cap \mathcal{O}_{12}^G$, it is clear that $\mathcal{E}_{12}^G$ and $\widetilde{\mathcal{D}}_{12}^G$ are complete lattices that are sublattices of $\mathcal{O}_{12}$. Likewise, $\mathcal{D}_{12}$ and $\widetilde{\mathcal{E}}_{12}$ are join-complete sublattices of $\mathcal{O}_{12}$, $\mathcal{D}_{12}^G = \mathcal{D}_{12} \cap \mathcal{O}_{12}^G$, and $\widetilde{\mathcal{E}}_{12}^G = \widetilde{\mathcal{E}}_{12} \cap \mathcal{O}_{12}^G$. Again it is clear that $\mathcal{D}_{12}^G$ and $\widetilde{\mathcal{E}}_{12}^G$ are complete lattices that are sublattices of $\mathcal{O}_{12}$.

(1) Let $\mathcal{B} \subset \mathcal{E}_{12}^G$ be arbitrary. Then for all $\eta \in \ell$ and $x \in \mathcal{L}_1$ we have

$$\tau_\eta\left((\inf_{\mathcal{O}_{12}} \mathcal{B})(x)\right) = \tau_\eta\left(\inf_{\mathcal{L}_2}\{\varepsilon(x) : \varepsilon \in \mathcal{B}\}\right) = \inf_{\mathcal{L}_2}\{\tau_\eta(\varepsilon(x)) : \varepsilon \in \mathcal{B}\} =$$

$$\inf_{\mathcal{L}_2}\{\varepsilon(\theta_\eta(x)) : \varepsilon \in \mathcal{B}\} = (\inf_{\mathcal{O}_{12}} \mathcal{B})(\theta_\eta(x)).$$

That is, the infimum of $\mathcal{B}$ in $\mathcal{O}_{12}$ is G-invariant. Thus $\mathcal{E}_{12}^G$ is a meet-complete sublattice of $\mathcal{O}_{12}$. The rest of the proof is similar.

Corollary 6 *Lemma 1, Proposition 3, and Proposition 22 now give the following results.*

1. *$\mathcal{E}_{12}^G$ and $\widetilde{\mathcal{D}}_{12}^G$ are meet-complete sublattices of $\mathcal{E}_{12}$ and $\widetilde{\mathcal{D}}_{12}$, respectively.*

2. *$\mathcal{D}_{12}^G$ and $\widetilde{\mathcal{E}}_{12}^G$ are join-complete sublattices of $\mathcal{D}_{12}$ and $\widetilde{\mathcal{E}}_{12}$, respectively.*

Definition 27 *We now adopt the following notation.*

1. *The set of G-***invariant** *$\psi \in \mathbf{\Lambda}_{12}$ will be denoted $\mathbf{\Lambda}_{12}^{G}$.*

2. *The set of G-***invariant** *$\psi \in \mathbf{M}_{12}$ will be denoted $\mathbf{M}_{12}^{G}$.*

3. *The set of G-***invariant** *$(\alpha, \beta) \in \mathcal{D}\widetilde{\mathcal{D}}_{21}$ will be denoted $\mathcal{D}\widetilde{\mathcal{D}}_{21}^{G}$.*

4. *The set of G-***invariant** *$(\alpha, \beta) \in \widetilde{\mathcal{E}}\mathcal{E}_{21}$ will be denoted $\widetilde{\mathcal{E}}\mathcal{E}_{21}^{G}$.*

The **sublattice properties of these sets** will now be estabilished.

Proposition 23 *To begin with, we have the following.*

1. *$\mathbf{\Lambda}_{12}^{G}$ is a meet-complete sublattice of $\mathcal{O}_{12}$.*

2. *$\mathbf{M}_{12}^{G}$ is a join-complete sublattice of $\mathcal{O}_{12}$.*

Proof (1) It is clear that $\mathbf{o}, \mathbf{e} \in \mathbf{\Lambda}_{12}^{G}$; moreover, if $\psi, \varphi \in \mathbf{\Lambda}_{12}^{G}$, it is also clear that $\psi \vee \varphi$ and $\psi \wedge \varphi$ are both sup-generating and G-invariant. If $\mathcal{B}$ is an arbitrary subset of $\mathbf{\Lambda}_{12}^{G}$, then $\inf_{\mathcal{O}_{12}} \mathcal{B}$ is sup-generating, by Proposition 6, and G-invariant, by Corollary 4.

(2) It is clear that $\mathbf{o}, \mathbf{e} \in \mathbf{M}_{12}^{G}$; moreover, if $\psi, \varphi \in \mathbf{M}_{12}^{G}$, it is also clear that $\psi \vee \varphi$ and $\psi \wedge \varphi$ are both inf-generating and G-invariant. If $\mathcal{B}$ is an arbitrary subset of $\mathbf{M}_{12}^{G}$, then $\sup_{\mathcal{O}_{12}} \mathcal{B}$ is inf-generating, by Proposition 6, and G-invariant, by Corollary 4.

Corollary 7 *Lemma 1, Proposition 6, and Proposition 22 now give the following results.*

1. *$\mathbf{\Lambda}_{12}^{G}$ is a meet-complete sublattice of $\mathbf{\Lambda}_{12}$.*

2. *$\mathbf{M}_{12}^{G}$ is a join-complete sublattice of $\mathbf{M}_{12}$.*

Lemma 2 *If $\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\}$ is a subset of $\mathcal{O}_{21} \times \mathcal{O}_{21}$, then the following hold.*

(a) $\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\} = \Big(\inf_{\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}, \sup_{\mathcal{O}_{21}}\{\beta_\nu : \nu \in \mathcal{N}\}\Big)$.

(b) $\inf_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\} = \Big(\sup_{\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}, \inf_{\mathcal{O}_{21}}\{\beta_\nu : \nu \in \mathcal{N}\}\Big)$.

(c) *If $\alpha_\nu \in \mathcal{D}_{21}^{G}$ $(\widetilde{\mathcal{E}}_{21}^{G})$ for all $\nu \in \mathcal{N}$, then so is $\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}$.*

(d) *If $\beta_\nu \in \mathcal{E}_{21}^{G}$ $(\widetilde{\mathcal{D}}_{21}^{G})$ for all $\nu \in \mathcal{N}$, then so is $\inf_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{\beta_\nu : \nu \in \mathcal{N}\}$.*

Let $\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\}$ be, in addition, a finite set. Then we have the following:

(e) *If $\alpha_\nu \in \mathcal{D}_{21}^{G}$ $(\widetilde{\mathcal{E}}_{21}^{G})$ for all $\nu \in \mathcal{N}$, then so is $\inf_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}$.*

(f) *If $\beta_\nu \in \mathcal{E}_{21}^{G}$ $(\widetilde{\mathcal{D}}_{21}^{G})$ for all $\nu \in \mathcal{N}$, then so is $\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}$.*

Proof (a) and (b) are immediate consequences of the last part of Definition 10; (c), (d), (e), and (f) follow from Corollary 6.

Lemma 3 *If $\{(\alpha_\nu, \beta_\nu)\}$ is a subset of $\mathbf{\Delta}_{21}$, then so are*

$$\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu)\} \textit{ and } \inf_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu)\}.$$

Proof The lemma is trivial if no $\alpha_\nu = \mathbf{E}$. Thus assume that $\alpha_\nu = \mathbf{E}$ and $\beta_\nu = \mathbf{O}$ for some $\nu \in \mathcal{N}$. Then $\inf_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\} = (\mathbf{E}, \mathbf{O}) \in \mathbf{\Delta}_{21}$; whereas for $\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\} = (\inf_{\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}, \sup_{\mathcal{O}_{21}}\{\beta_\nu : \nu \in \mathcal{N}\})$ we may either omit the $\beta_\nu = \mathbf{O}$ from $\sup_{\mathcal{O}_{21}}\{\beta_\nu : \nu \in \mathcal{N}\}$ and the corresponding α_ν from $\inf_{\mathcal{O}_{21}}\{\alpha_\nu : \nu \in \mathcal{N}\}$, in which case it is clear that $\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\} \in \mathbf{\Delta}_{21}$, or have that $\sup_{\mathcal{O}_{21}\times\mathcal{O}_{21}}\{(\alpha_\nu, \beta_\nu) : \nu \in \mathcal{N}\} = (\mathbf{E}, \mathbf{O}) \in \mathbf{\Delta}_{21}$. This completes the proof.

Lemma 4 *$\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$ is a sublattice of $(\mathcal{D}\widetilde{\mathcal{D}}_{21}, \leq)$ and $\widetilde{\mathcal{E}}\mathcal{E}_{21}^G$ is a sublattice of $(\widetilde{\mathcal{E}}\mathcal{E}_{21}, \leq)$.*

Proof $(\mathcal{D}\widetilde{\mathcal{D}}_{21}, \leq)$ is a complete lattice whose least and greatest elements are $(\mathbf{E}, \mathbf{O})$ and $(\mathbf{O}, \mathbf{E})$, which are plainly G-invariant. According to Lemma 2, if $\{(\alpha_i, \beta_i)\}$ is a finite subset of either $\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$ or $\widetilde{\mathcal{E}}\mathcal{E}_{21}^G$, then so are

$$(\inf_{\mathcal{O}_{21}}\{\alpha_i\}, \sup_{\mathcal{O}_{21}}\{\beta_i\}) \text{ and } (\sup_{\mathcal{O}_{21}}\{\alpha_i\}, \inf_{\mathcal{O}_{21}}\{\beta_i\}).$$

The lemma can thus be seen to follow from Lemma 3.

We are now in a position to prove the G**-invariant analog of Theorem 1**.

Theorem 4 *The set of G-invariant morphological connections between $\mathcal{L}_1$ and $\mathcal{L}_2$ is the graph of the lattice isomorphism between $\mathbf{\Lambda}_{12}^G$ and $\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$ given for all $\psi \in \mathbf{\Lambda}_{12}^G$ and for all $(\varsigma, \tilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}_{21}^G$ by $\psi \longmapsto (\underline{\cdot\psi}, \overline{\cdot\psi})$ and $(\varsigma, \tilde{\varsigma}) \longmapsto \overline{\varsigma\tilde{\varsigma}}$. Moreover, $\psi \longmapsto (\underline{\psi\cdot}, \overline{\psi\cdot})$ is a lattice isomorphism of $\mathbf{M}_{12}^G$ onto $\widetilde{\mathcal{E}}\mathcal{E}_{21}^G$ whose inverse is given by $(\tilde{\epsilon}, \epsilon) \longmapsto \underline{\tilde{\epsilon}\epsilon}$.*

Proof Theorem 1 and Theorem 3 show that $\psi \longmapsto (\underline{\cdot\psi}, \overline{\cdot\psi})$ is a bijection of $\mathbf{\Lambda}_{12}^G$ onto $\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$. Theorem 1 also shows that if $\psi, \varphi \in \mathbf{\Lambda}_{12}^G$, then

$$\psi \wedge \varphi \longmapsto \inf_{\mathcal{D}\widetilde{\mathcal{D}}_{21}}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\} \text{ and } \psi \vee \varphi \longmapsto \sup_{\mathcal{D}\widetilde{\mathcal{D}}_{21}}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\}.$$

Moreover, since $\psi \wedge \varphi, \psi \vee \varphi \in \mathbf{\Lambda}_{12}^G$, it follows as well from Theorem 1 that

$$\inf_{\mathcal{D}\widetilde{\mathcal{D}}_{21}}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\} \text{ and } \sup_{\mathcal{D}\widetilde{\mathcal{D}}_{21}}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\}$$

are elements of $\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$. Since $(\underline{\cdot\psi}, \overline{\cdot\psi})$ and $(\underline{\cdot\varphi}, \overline{\cdot\varphi})$ are also elements of $\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$, and since $\mathcal{D}\widetilde{\mathcal{D}}_{21}^G$ is a sublattice of $(\mathcal{D}\widetilde{\mathcal{D}}_{21}, \leq)$, it follows that

$$\inf_{\mathcal{D}\widetilde{\mathcal{D}}_{21}}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\} = \inf_{\mathcal{D}\widetilde{\mathcal{D}}_{21}^G}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\}$$

and

$$\sup_{\mathcal{D}\widetilde{\mathcal{D}}_{21}}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\} = \sup_{\mathcal{D}\widetilde{\mathcal{D}}_{21}^G}\{(\underline{\cdot\psi}, \overline{\cdot\psi}), (\underline{\cdot\varphi}, \overline{\cdot\varphi})\}.$$

This shows that $\psi \longmapsto (\underline{\cdot\psi}, \overline{\cdot\psi})$ is a lattice isomorphism of $(\mathbf{\Lambda}_{12}^G, \preceq_p)$ onto $(\mathcal{D}\widetilde{\mathcal{D}}_{21}^G, \leq)$. The proof that $\psi \longmapsto (\underline{\psi\cdot}, \overline{\psi\cdot})$ is a lattice isomorphism of $\mathbf{M}_{12}^G$ onto $\widetilde{\mathcal{E}}\mathcal{E}_{21}^G$ is similar.

Corollary 8 *The following are now immediate consequences of Corollary 7.*

1. $\mathcal{D}\widetilde{\mathcal{D}}^G_{21}$ *is a meet-complete sublattice of* $(\mathcal{D}\widetilde{\mathcal{D}}_{21}, \leq)$.

2. $\widetilde{\mathcal{E}}\mathcal{E}^G_{21}$ *is a join-complete sublattice of* $(\widetilde{\mathcal{E}}\mathcal{E}_{21}, \leq)$.

Corollary 9 *The* G**-invariant versions of the Achache-Serra propositions** *are*

1. *The set of* G*-invariant Galois connections between* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *is the graph of the lattice isomorphism between* $\widetilde{\mathcal{D}}^G_{12}$ *and* $\widetilde{\mathcal{D}}^G_{21}$ *given for all* $\widetilde{\delta} \in \widetilde{\mathcal{D}}^G_{12}$ *and* $\widetilde{\varsigma} \in \widetilde{\mathcal{D}}^G_{21}$*, respectively, by* $\widetilde{\delta} \longmapsto \overline{\cdot\widetilde{\delta}}$ *and* $\widetilde{\varsigma} \longmapsto \overline{\mathbf{O}\widetilde{\varsigma}}$.

2. *The set of* G*-adjunctions between* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *is the graph of the dual-lattice isomorphism between* $\mathcal{E}^G_{12}$ *and* $\mathcal{D}^G_{21}$ *given for all* $\varepsilon \in \mathcal{E}^G_{12}$ *and* $\varsigma \in \mathcal{D}^G_{21}$ *by* $\varepsilon \longmapsto \underline{\cdot\varepsilon}$ *and* $\varsigma \longmapsto \overline{\varsigma\mathbf{E}}$.

The G**-invariant version of Proposition 12** is as follows.

Proposition 24 *If* $\psi \in \mathcal{O}^G_{12}$*, then we have that*

(a) $\psi \in \mathbf{\Lambda}^G_{12}$ *if and only if there is an* $\varepsilon \in \mathcal{E}^G_{12}$ *and a* $\widetilde{\delta} \in \widetilde{\mathcal{D}}^G_{12}$ *such that* $\psi = \varepsilon \wedge \widetilde{\delta}$.

(b) $\psi \in \mathbf{M}^G_{12}$ *if and only if there is a* $\delta \in \mathcal{D}^G_{12}$ *and an* $\widetilde{\varepsilon} \in \widetilde{\mathcal{E}}^G_{12}$ *such that* $\psi = \delta \vee \widetilde{\varepsilon}$.

As a consequence of these results we obtain, in particular, the following results.

(c) *If* $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}^G_{21}$*, then* $\overline{\varsigma\widetilde{\varsigma}} = \overline{\varsigma\mathbf{E}} \wedge \overline{\mathbf{O}\widetilde{\varsigma}}$.

(d) *If* $(\widetilde{\epsilon}, \epsilon) \in \widetilde{\mathcal{E}}\mathcal{E}^G_{21}$*, then* $\underline{\widetilde{\epsilon}\epsilon} = \underline{\widetilde{\epsilon}\mathbf{E}} \vee \underline{\mathbf{O}\epsilon}$.

Proof (a) If there is an $\varepsilon \in \mathcal{E}^G_{12}$ and a $\widetilde{\delta} \in \widetilde{\mathcal{D}}^G_{12}$ such that $\psi = \varepsilon \wedge \widetilde{\delta}$, then by Proposition 12 and the fact that $\varepsilon \wedge \widetilde{\delta}$ is G-invariant, we see that $\psi \in \mathbf{\Lambda}^G_{12}$. On the other hand, if $\psi \in \mathbf{\Lambda}^G_{12}$, then by Proposition 12 we see that there is an $\varepsilon \in \mathcal{E}_{12}$ and a $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}$ such that $\psi = \varepsilon \wedge \widetilde{\delta}$; in fact, if we put $\varsigma = \underline{\cdot\psi}$ and $\widetilde{\varsigma} = \overline{\cdot\psi}$, then a suitable ε is $\overline{\varsigma\mathbf{E}}$ and a suitable $\widetilde{\delta}$ is $\overline{\mathbf{O}\widetilde{\varsigma}}$. Since ς and $\widetilde{\varsigma}$ are G-invariant, (a) is proved.

(b) If there is a $\delta \in \mathcal{D}^G_{12}$ and an $\widetilde{\varepsilon} \in \widetilde{\mathcal{E}}^G_{12}$ such that $\psi = \delta \vee \widetilde{\varepsilon}$, then by Proposition 12 and the fact that $\delta \vee \widetilde{\varepsilon}$ is G-invariant, we see that $\psi \in \mathbf{M}^G_{12}$. On the other hand, if $\psi \in \mathbf{M}^G_{12}$, then by Proposition 12 we see that there is a $\delta \in \mathcal{D}_{12}$ and an $\widetilde{\varepsilon} \in \widetilde{\mathcal{E}}_{12}$ such that $\psi = \delta \vee \widetilde{\varepsilon}$; in fact, if we put $\widetilde{\epsilon} = \underline{\psi\cdot}$ and $\epsilon = \overline{\psi\cdot}$, then a suitable δ is $\underline{\widetilde{\epsilon}\mathbf{E}}$ and a suitable $\widetilde{\varepsilon}$ is $\underline{\mathbf{O}\epsilon}$. Since $\widetilde{\epsilon}$ and ϵ are G-invariant, (b) is proved.

(c) If $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}^G_{21}$, then by Theorem 4, $\overline{\varsigma\widetilde{\varsigma}} \in \mathbf{\Lambda}^G_{12}$; hence, by (a) and its proof it therefore follows that $\overline{\varsigma\widetilde{\varsigma}} = \overline{\varsigma\mathbf{E}} \wedge \overline{\mathbf{O}\widetilde{\varsigma}}$.

(d) If $(\widetilde{\epsilon}, \epsilon) \in \widetilde{\mathcal{E}}\mathcal{E}^G_{21}$, then by Theorem 4, $\underline{\widetilde{\epsilon}\epsilon} \in \mathbf{M}^G_{12}$; hence, by (b) and its proof it therefore follows that $\underline{\widetilde{\epsilon}\epsilon} = \underline{\widetilde{\epsilon}\mathbf{E}} \vee \underline{\mathbf{O}\epsilon}$.

3.3 Interim Summary

Thus far I have revisited most of the earlier results for general lattice mappings (those contained in sections 2.1 through 2.7) and established analogous results for G-invariant mappings where appropriate. We are at the point in the earlier general development when the next goal was the establishment of the decomposition theorem, Theorem 2. This goal was reached via Proposition 13 (which states that $\mathbf{\Lambda}_{12}$ and $\mathbf{M}_{12}$ are, respectively, sup- and inf-generating subsets of $\mathcal{O}_{12}$) and the following result (which was not explicitly stated).

Lemma 5 *If $\psi \in \mathcal{O}_{12}$, then we have the following.*

1. $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}_{21}$ *and* $[\varsigma, \widetilde{\varsigma}] \ll \cdot\mathcal{K}(\psi)$ *if and only if* $\overline{\overline{\varsigma\widetilde{\varsigma}}} \in \mathbf{\Lambda}_{12}$ *and* $\overline{\overline{\varsigma\widetilde{\varsigma}}} \preceq_p \psi$.
2. $(\widetilde{\epsilon}, \epsilon) \in \widetilde{\mathcal{E}}\mathcal{E}_{21}$ *and* $[\widetilde{\epsilon}, \epsilon] \ll \mathcal{K} \cdot (\psi)$ *if and only if* $\underline{\underline{\widetilde{\epsilon}\epsilon}} \in \mathbf{M}_{12}$ *and* $\psi \preceq_p \underline{\underline{\widetilde{\epsilon}\epsilon}}$.

In other words, the conditions in (a) and (b) of Theorem 2 that define the two sets of mappings that respectively constitute the theorem's supremum and infimum decompositions, are respectively equivalent to $\{\lambda \in \mathbf{\Lambda}_{12} : \lambda \preceq_p \psi\}$ and $\{\mu \in \mathbf{M}_{12} : \psi \preceq_p \mu\}$, i.e., are just the sup- and inf-generating representations furnished by $\mathbf{\Lambda}_{12}$ and $\mathbf{M}_{12}$. The establishment of the following G-invariant analog of Lemma 5 is quite straightfoward. The same cannot be said of Proposition 13, however; indeed, its G-invariant analog turns out to be false.

Lemma 6 *If $\psi \in \mathcal{O}^G_{12}$, then we have the following.*

1. $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}^G_{21}$ *and* $[\varsigma, \widetilde{\varsigma}] \ll \cdot\mathcal{K}(\psi)$ *if and only if* $\overline{\overline{\varsigma\widetilde{\varsigma}}} \in \mathbf{\Lambda}^G_{12}$ *and* $\overline{\overline{\varsigma\widetilde{\varsigma}}} \preceq_p \psi$.
2. $(\widetilde{\epsilon}, \epsilon) \in \widetilde{\mathcal{E}}\mathcal{E}^G_{21}$ *and* $[\widetilde{\epsilon}, \epsilon] \ll \mathcal{K} \cdot (\psi)$ *if and only if* $\underline{\underline{\widetilde{\epsilon}\epsilon}} \in \mathbf{M}^G_{12}$ *and* $\psi \preceq_p \underline{\underline{\widetilde{\epsilon}\epsilon}}$.

Proof (1) By Theorem 4, $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}^G_{21}$ is equivalent to $\lambda \equiv \overline{\overline{\varsigma\widetilde{\varsigma}}} \in \mathbf{\Lambda}^G_{12}$, which in turn is equivalent to $\cdot\mathcal{K}(\lambda)$ is an interval function; also, $(\varsigma, \widetilde{\varsigma}) \in \mathcal{D}\widetilde{\mathcal{D}}^G_{21}$ is equivalent to

$$[\varsigma(y), \widetilde{\varsigma}(y)] = [\underline{\cdot\lambda}(y), \overline{\cdot\lambda}(y)] \quad \forall y \in \mathcal{L}_2.$$

Now, by definition we have the following:

(a) $\underline{\cdot\lambda}(y) = \inf\{x \in \mathcal{L}_1 : y \preceq \lambda(x)\} = \inf(\cdot\mathcal{K}(\lambda)(y))$.
(b) $\overline{\cdot\lambda}(y) = \sup\{x \in \mathcal{L}_1 : y \preceq \lambda(x)\} = \sup(\cdot\mathcal{K}(\lambda)(y))$.

Hence $[\varsigma, \widetilde{\varsigma}]$ is the interval function $\cdot\mathcal{K}(\lambda)$, i.e., $[\varsigma, \widetilde{\varsigma}] = \cdot\mathcal{K}(\lambda)$. Thus $[\varsigma, \widetilde{\varsigma}] \ll \cdot\mathcal{K}(\psi)$ can be written alternatively as $\cdot\mathcal{K}(\lambda) \ll \cdot\mathcal{K}(\psi)$, which by Proposition 14 is equivalent to $\lambda \preceq_p \psi$. This proves (1). The proof of (2) is similar.

The obvious G-invariant analog of Proposition 13 is that $\mathbf{\Lambda}^G_{12}$ **and** $\mathbf{M}^G_{12}$ **are, respectively, sup- and inf-generating subsets of** $\mathcal{O}^G_{12}$**. This proposition is not generally true, however**. Indeed, for $\mathbf{\Lambda}^G_{12}$, while there are always enough mappings in $\mathbf{\Lambda}_{12}$ to sup-decompose an arbitrary mapping in $\mathcal{O}_{12}$, **in some concrete instances**—and examples will be furnished—**there are not enough mappings in** $\mathbf{\Lambda}^G_{12}$ **to sup-decompose an arbitrary mapping in** $\mathcal{O}^G_{12}$. In order to obtain the tools needed to get to the root of this insufficiency, we now take up the development of the kernel theory of G-invariant mappings, a theory of considerable interest in its own right.

3.4 Left-Kernel Theory of G-Invariant Mappings

First, I establish some elementary G-invariance properties of left-kernels.

Definition 28 *Two further types of G-invariance are the following.*

1. $\mathcal{F} : \mathcal{L}_2 \longrightarrow \mathcal{P}(\mathcal{L}_1)$ *is* **G-invariant** *if* $\theta_\eta(\mathcal{F}(y)) = \mathcal{F}(\tau_\eta(y))$ *for all* $\eta \in \ell$ *and* $y \in \mathcal{L}_2$.

2. $F : \mathcal{O}_{12} \longrightarrow \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *is* **G-invariant** *if* $F(\psi) : \mathcal{L}_2 \longrightarrow \mathcal{P}(\mathcal{L}_1)$ *is G-invariant* $\forall\ \psi \in \mathcal{O}_{12}^G$.

Note that $\theta_\eta(\emptyset)$ is understood to be $\emptyset$. Thus, if in reference to (1) above we have that $\mathcal{F}$ is G-invariant and $\mathcal{F}(y) = \emptyset$, then $\mathcal{F}(\tau_\eta(y)) = \emptyset$ for all $\eta \in \ell$.

Proposition 25 $\cdot\mathcal{K} : \mathcal{O}_{12}^G \longrightarrow \mathcal{P}(\mathcal{L}_1)^{\mathcal{L}_2}$ *is G-invariant.*

Proof By definition, $\theta_\eta(\cdot\mathcal{K}(\psi)(y)) = \{\theta_\eta(x) : x \in \mathcal{L}_1, y \preceq \psi(x)\}$. If ψ is G-invariant, then $y \preceq \psi(x) \iff \tau_\eta(y) \preceq \psi(\theta_\eta(x))$. Hence,

$$\theta_\eta(\cdot\mathcal{K}(\psi)(y)) = \{\theta_\eta(x) : \tau_\eta(y) \preceq \psi(\theta_\eta(x))\} = \cdot\mathcal{K}(\psi)(\tau_\eta(y)).$$

Corollary 10 *If* $\psi \in \mathcal{O}_{12}^G$, $\xi \in \ell$, *and* $\cdot\mathcal{K}(\psi)(\xi) = \emptyset$, *then* $\cdot\mathcal{K}(\psi)(\eta) = \emptyset$ *for all* $\eta \in \ell$; *in fact,* $\cdot\mathcal{K}(\psi)(y) = \emptyset$ *for all* $y \in \mathcal{L}_2 \setminus \{o\}$.

Proof Let $\mu = \eta - \xi$. Then $\emptyset = \theta_\mu(\cdot\mathcal{K}(\psi)(\xi)) = \cdot\mathcal{K}(\psi)(\tau_{\eta-\xi}(\xi)) = \cdot\mathcal{K}(\psi)(\eta)$. Since the function $\cdot\mathcal{K}(\psi)(\cdot)$ is decreasing, it follows that $\cdot\mathcal{K}(\psi)(y) = \emptyset$ for all $y \in [\eta, e]$ for all $\eta \in \ell$. Moreover, for all $y \in \mathcal{L}_2 \setminus \{o\}$, there is an $\eta \in \ell$ such that $\eta \preceq y$.

Corollary 11 *If* $\psi \in \mathcal{O}_{12}^G$ *and* $\eta \in \ell$, *then* $\cdot\mathcal{K}(\psi)(\eta) = \theta_\eta(\cdot\mathcal{K}(\psi)(\rho))$.

Lemma 7 *If* $\psi \in \mathcal{O}_{12}^G$, $y \in \mathcal{L}_2$, *and* $\cdot\mathcal{K}(\psi)(\rho) \neq \emptyset$, *then*

$$\cdot\mathcal{K}(\psi)(y) = \bigcap\{\theta_\eta(\cdot\mathcal{K}(\psi)(\rho)) : \eta \in \ell, \eta \preceq y\}.$$

Proof By definition, $\cdot\mathcal{K}(\psi)(y) = \{x \in \mathcal{L}_1 : \sup\{\eta \in \ell : \eta \preceq y\} \preceq \psi(x)\}$. Moreover, it is clear that $\{x \in \mathcal{L}_1 : \sup\{\eta \in \ell : \eta \preceq y\} \preceq \psi(x)\} = \{x \in \mathcal{L}_1 : \eta \preceq \psi(x)\ \forall\ \eta \preceq y \text{ in } \ell\}$. Therefore, $\cdot\mathcal{K}(\psi)(y) = \bigcap\{\cdot\mathcal{K}(\psi)(\eta) : \eta \in \ell, \eta \preceq y\} = \bigcap\{\theta_\eta(\cdot\mathcal{K}(\psi)(\rho)) : \eta \in \ell, \eta \preceq y\}$.

Proposition 26 *If* $\varphi, \psi \in \mathcal{O}_{12}^G$, *then we have the following.*

1. $\cdot\mathcal{K}(\varphi)(\rho) \subset \cdot\mathcal{K}(\psi)(\rho) \Longrightarrow \cdot\mathcal{K}(\varphi)(\eta) \subset \cdot\mathcal{K}(\psi)(\eta)$ *for all* $\eta \in \ell$.

2. $\cdot\mathcal{K}(\varphi) \ll \cdot\mathcal{K}(\psi) \iff \cdot\mathcal{K}(\varphi)(\rho) \subset \cdot\mathcal{K}(\psi)(\rho)$.

Proof (1) Since $\theta_\eta(\cdot\mathcal{K}(\varphi)(\rho)) \subset \theta_\eta(\cdot\mathcal{K}(\psi)(\rho))$, it follows that $\cdot\mathcal{K}(\varphi)(\tau_\eta(\rho)) \subset \cdot\mathcal{K}(\psi)(\tau_\eta(\rho))$. Since $\tau_\eta(\rho) = \eta$, the proof of (1) is complete.

(2) Denote the set $\{\eta \in \ell : \eta \preceq y\}$ by $\ell(y)$. If $y \in \mathcal{L}_2$, then

$$\cdot\mathcal{K}(\varphi)(y) = \bigcap_{\eta\in\ell(y)} \theta_\eta(\cdot\mathcal{K}(\varphi)(\rho)) \quad \text{and} \quad \cdot\mathcal{K}(\psi)(y) = \bigcap_{\eta\in\ell(y)} \theta_\eta(\cdot\mathcal{K}(\psi)(\rho)).$$

Therefore $\cdot\mathcal{K}(\varphi)(y) = \bigcap_{\eta\in\ell(y)} \cdot\mathcal{K}(\varphi)(\eta) \subset \bigcap_{\eta\in\ell(y)} \cdot\mathcal{K}(\psi)(\eta) =$

$$\bigcap_{\eta\in\ell(y)} \theta_\eta(\cdot\mathcal{K}(\psi)(\rho)) = \cdot\mathcal{K}(\psi)(y).$$

Corollary 12 *If $\varphi, \psi \in \mathcal{O}_{12}^G$, then $\varphi \preceq_p \psi \iff \cdot\mathcal{K}(\varphi)(\rho) \subset \cdot\mathcal{K}(\psi)(\rho)$.*

3.4.1 Ronse's Theorem and Minimal Bases

Next, I introduce several concepts and results from the Heijmans-Ronse-Serra theory.

Definition 29 *If $a, b \in \mathcal{L}_1$, then for all $x \in \mathcal{L}_1$ we* **define the mapping** *$\sigma_{ab} \in \mathcal{O}_{12}$ by*

$$\sigma_{ab}(x) = \bigvee\{\eta \in \ell : \theta_\eta(a) \preceq x \preceq \theta_\eta(b)\}.$$

Remark 13 *If $a \not\preceq b$, then $\sigma_{ab}(x) = o$ for all $x \in \mathcal{L}_1$.*

Lemma 8 *$\sigma_{ab} \in \mathcal{O}_{12}^G$ for all $a, b \in \mathcal{L}_1$.*

Proof We show for each $\mu \in \ell$ and $x \in \mathcal{L}_1$ that

$$\tau_\mu(\sigma_{ab}(x)) = \bigvee\{\eta \in \ell : \theta_\eta(a) \preceq \theta_\mu(x) \preceq \theta_\eta(b)\} = \sigma_{ab}(\theta_\mu(x)).$$

First of all, we have $\tau_\mu(\sigma_{ab}(x)) = \bigvee\{\xi+\mu : \xi \in \ell, \theta_\xi(a) \preceq x \preceq \theta_\xi(b)\}$. Putting $\xi+\mu = \eta$ we obtain $\bigvee\{\xi + \mu : \xi \in \ell, \theta_\xi(a) \preceq x \preceq \theta_\xi(b)\} =$

$$\bigvee\{\eta \in \ell : \theta_{\eta-\mu}(a) \preceq x \preceq \theta_{\eta-\mu}(b)\} = \bigvee\{\eta \in \ell : \theta_\eta(a) \preceq \theta_\mu(x) \preceq \theta_\eta(b)\},$$

where $\theta_{\eta-\mu}(a) \preceq x \preceq \theta_{\eta-\mu}(b) \iff \theta_\eta(a) \preceq \theta_\mu(x) \preceq \theta_\eta(b)$ validates the last step.

Definition 30 *The* **bi-kernel** *$\mathcal{W}(\psi)$ of a mapping $\psi \in \mathcal{O}_{12}^G$ is defined by*

$$\mathcal{W}(\psi) = \{(a, b) \in \mathcal{L}_1 \times \mathcal{L}_1 : a \preceq x \preceq b \Longrightarrow \rho \preceq \psi(x)\}.$$

Remark 14 *If $\psi \in \mathcal{O}_{12}^G$, then*

$$\mathcal{W}(\psi) = \{(a, b) \in \mathcal{L}_1 \times \mathcal{L}_1 : [a, b] \subset \cdot\mathcal{K}(\psi)(\rho)\}.$$

Consequently, if $\psi, \varphi \in \mathcal{O}_{12}^G$ and $\cdot\mathcal{K}(\psi)(\rho) = \cdot\mathcal{K}(\varphi)(\rho)$, then $\mathcal{W}(\psi) = \mathcal{W}(\varphi)$.

The next theorem, and its proof, were provided to me by C. Ronse in a private communication. Its resemblance to part (a) of the Banon-Barrera decomposition theorem (Thm. 2) should not go unnoticed.

Theorem 5 (Ronse) *If $\psi \in \mathcal{O}_{12}^G$, then $\psi(x) = \bigvee\{\sigma_{ab}(x) : (a,b) \in \mathcal{W}(\psi)\}$ for all $x \in \mathcal{L}_1$.*

Proof For each $x \in \mathcal{L}_1$ let $\varphi(x)$ be given by $\varphi(x) = \bigvee\{\sigma_{ab}(x) : (a,b) \in \mathcal{W}(\psi)\}$. First we show that $\varphi(x) \preceq \psi(x)$ for all $x \in \mathcal{L}_1$. If $\eta \in \ell$ and $(a,b) \in \mathcal{W}(\psi)$, then

$$\theta_\eta(a) \preceq x \preceq \theta_\eta(b) \Longrightarrow a \preceq \theta_\eta^{-1}(x) \preceq b \Longrightarrow \rho \preceq \psi(\theta_\eta^{-1}(x)) = \tau_\eta^{-1}(\psi(x)).$$

Hence $\eta = \tau_\eta(\rho) \preceq \psi(x)$ and it follows that

$$\sigma_{ab}(x) = \bigvee\{\eta \in \ell : \theta_\eta(a) \preceq x \preceq \theta_\eta(b)\} \preceq \psi(x).$$

Thus $\varphi(x) = \bigvee\{\sigma_{ab}(x) : (a,b) \in \mathcal{W}(\psi)\} \preceq \psi(x)$ for all $x \in \mathcal{L}_1$.

Now we show that $\psi(x) \preceq \varphi(x)$ for all $x \in \mathcal{L}_1$. If $\eta \in \ell$ and $\eta \preceq \psi(x)$, then

$$\rho = \tau_\eta^{-1}(\eta) \preceq \tau_\eta^{-1}(\psi(x)) \preceq \psi(\theta_\eta^{-1}(x)).$$

Now if $x' \in \mathcal{L}_1$ and $\theta_\eta^{-1}(x) \preceq x' \preceq \theta_\eta^{-1}(x)$, then $x' = \theta_\eta^{-1}(x)$, which in turn implies that $\rho \preceq \psi(x')$. Hence $(\theta_\eta^{-1}(x), \theta_\eta^{-1}(x)) \in \mathcal{W}(\psi)$. Since $\theta_\eta(\theta_\eta^{-1}(x)) \preceq x \preceq \theta_\eta(\theta_\eta^{-1}(x))$, it follows that $\eta \preceq \bigvee\{\xi \in \ell : \theta_\xi(\theta_\eta^{-1}(x)) \preceq x \preceq \theta_\xi(\theta_\eta^{-1}(x))\} = \sigma_{(\theta_{-\eta}(x))(\theta_{-\eta}(x))}(x)$. Hence,

$$\eta \preceq \bigvee\{\sigma_{ab}(x) : (a,b) \in \mathcal{W}(\psi)\} = \varphi(x),$$

and, since ℓ is sup-generating, we see that $\psi(x) \preceq \varphi(x)$ for all $x \in \mathcal{L}_1$. Thus $\psi = \varphi$.

Corollary 13 *Applying Remark 14 to Ronse's theorem we obtain the following.*

1. *Each $\psi \in \mathcal{O}_{12}^G$ can be decomposed as $\psi = \sup_{\mathcal{O}_{12}^G}\{\sigma_{ab} : [a,b] \subset \cdot\mathcal{K}(\psi)(\rho)\}$.*

2. *If $\varphi, \psi \in \mathcal{O}_{12}^G$ have the same bi-kernel, then $\varphi = \psi$.*

I call the formula

$$\psi = \sup_{\mathcal{O}_{12}^G}\{\sigma_{ab} : [a,b] \subset \cdot\mathcal{K}(\psi)(\rho)\},$$

or its equivalent, a **Ronse decomposition** of ψ. Henceforth **we write $\mathcal{V}(\psi)$ for $\cdot\mathcal{K}(\psi)(\rho)$ and call $\mathcal{V}(\psi)$ the kernel of ψ.**

Remark 15 *If $\varphi, \psi \in \mathcal{O}_{12}^G$ and $\mathcal{V}(\varphi) = \mathcal{V}(\psi)$, then $\mathcal{W}(\varphi) = \mathcal{W}(\psi)$ and hence $\varphi = \psi$; that is, a G-invariant mapping is uniquely determined by its kernel, as well as by its bi-kernel.*

For more economical forms of Ronse's decomposition, we introduce the following definition. (This definition was introduced in an earlier article by Banon and Barrera [11].)

Definition 31 *If $\psi \in \mathcal{O}_{12}^G$, then a set $\mathcal{B}$ of closed intervals $[a,b] \subset \mathcal{V}(\psi)$ is called* **a basis for** ψ *if every closed interval contained in $\mathcal{V}(\psi)$ is contained in an interval of $\mathcal{B}$; we denote the set of all bases for ψ by $\mathbf{B}(\psi)$.*

Remark 16 *Define $\mathcal{B}^\star = \{[a,b] : [a,b] \subset \mathcal{V}(\psi)\}$. Then we have the following.*

1. *If $\mathcal{B} \in \mathbf{B}(\psi)$, then $\mathcal{B} \subset \mathcal{B}^\star$.*
2. *$\mathcal{B}^\star = \bigcup \mathbf{B}(\psi)$.*
3. *$\mathcal{B}^\star \in \mathbf{B}(\psi)$.*

$\mathcal{B}^\star$ **is thus the largest basis for** ψ; indeed, the Ronse decomposition given by part (1) of Corollary 13 is precisely $\psi = \sup_{\mathcal{O}_{12}^G}\{\sigma_{ab} : [a,b] \in \mathcal{B}^\star\}$.

Remark 17 *If $\psi \in \mathcal{O}_{12}^G$ and $\mathcal{B} \in \mathbf{B}(\psi)$, then $\psi = \sup\{\sigma_{ab} : [a,b] \in \mathcal{B}\}$; such a formula will also be called a Ronse decomposition of ψ. We distinguish the various Ronse decompositions by calling* $\sup\{\sigma_{ab} : [a,b] \in \mathcal{B}\}$ *the* **Ronse decomposition of ψ relative to the basis $\mathcal{B}$.**

There may also be a **smallest** or **minimal basis for** ψ, i.e., a basis $\mathcal{B}_\star$ such that $\mathcal{B}$ is not a proper subset of $\mathcal{B}_\star$ for all $\mathcal{B} \in \mathbf{B}(\psi)$. If there is such a basis, then the Ronse decomposition $\sup\{\sigma_{ab} : [a,b] \in \mathcal{B}_\star\}$ is minimal in the sense that it is relative to the smallest basis. The next few results somewhat illucidate the minimal basis question.

Remark 18 *If $\bigcap \mathbf{B}(\psi)$ is not empty and is also a member of $\mathbf{B}(\psi)$, then there is a smallest basis $\mathcal{B}_\star$ for ψ and $\mathcal{B}_\star = \bigcap \mathbf{B}(\psi)$.*

For the two lemmas that follow, I will use $\subseteq$ to denote *subset* and $\subset$ to denote *proper subset.* Since $(\mathcal{B}^\star, \subseteq)$ is a poset, we can ask whether it has maximal elements, i.e., whether there exist closed intervals $\imath \in \mathcal{B}^\star$ such that $\imath \not\subset \jmath$ for all $\jmath \in \mathcal{B}^\star$.

Lemma 9 *If $\psi \in \mathcal{O}_{12}^G$ and if every totally ordered subset of (chain in) $(\mathcal{B}^\star, \subseteq)$ has an upper bound in $\mathcal{B}^\star$, then $\mathcal{B}_\star$ exists and is equal to $\bigcap \mathbf{B}(\psi)$.*

Proof By Zorn's lemma, $(\mathcal{B}^\star, \subseteq)$ has maximal elements, the full set of which we denote $\mathcal{M}$. It follows that $\mathcal{M} \subset \bigcap \mathbf{B}(\psi)$, because the maximal closed-interval subsets of $\mathcal{V}(\psi)$ clearly must be in every basis for ψ. In addition, if $[a,b] \in \mathcal{B}^\star$, then by Hausdorff's maximal principle there is a maximal chain $\widehat{\mathcal{C}}$ in $\mathcal{B}^\star$ such that $[a,b] \in \widehat{\mathcal{C}}$. Let $[\alpha,\beta]$ be an upper bound of $\widehat{\mathcal{C}}$ in $\mathcal{B}^\star$. Then

$$[\alpha,\beta] \in \widehat{\mathcal{C}},\ [a,b] \subseteq [\alpha,\beta], \text{ and } [\alpha,\beta] \text{ is a maximal element of } \mathcal{B}^\star;$$

indeed, if there were a $\jmath \in \mathcal{B}^\star$ such that $[\alpha,\beta] \subset \jmath$, then $\widehat{\mathcal{C}}$ would not be maximal. Since it has now been shown that $\mathcal{M}$ is a basis for ψ, we have at once that $\mathcal{M} \in \mathbf{B}(\psi)$ and $\mathcal{M} \subset \bigcap \mathbf{B}(\psi)$, which clearly implies that $\mathcal{M} = \bigcap \mathbf{B}(\psi)$. The lemma is now an immediate consequence of Remark 18.

Definition 32 *$\mathcal{A} \subset \mathcal{L}_1$ is said to be* **closed under monotone limits** *if the following hold.*

1. *$\{u_\alpha\}$ is a net in $\mathcal{A}$ such that $u_\alpha \downarrow u = \inf\{u_\alpha\} \Longrightarrow u \in \mathcal{A}$.*
2. *$\{v_\alpha\}$ is a net in $\mathcal{A}$ such that $v_\alpha \uparrow v = \sup\{v_\alpha\} \Longrightarrow v \in \mathcal{A}$.*

Lemma 10 *If $\psi \in \mathcal{O}_{12}^G$ and if the kernel of ψ is closed under monotone limits, then every chain in $(\mathcal{B}^\star, \subseteq)$ has an upper bound in $\mathcal{B}^\star$.*

Proof Let $\mathcal{C}$ be an arbitrary chain in $(\mathcal{B}^\star, \subseteq)$. For each $\jmath \in \mathcal{C}$, let $a_\jmath = \inf \jmath$ and $b_\jmath = \sup \jmath$, so that $\jmath = [a_\jmath, b_\jmath]$. It is clear that $\mathcal{C} \subset [\inf_\jmath a_\jmath, \sup_\jmath b_\jmath]$. Moreover, since $\mathcal{V}(\psi)$ is closed under monotone limits, it follows that $\inf_\jmath a_\jmath$ and $\sup_\jmath b_\jmath$ lie in $\mathcal{V}(\psi)$. We therefore have that $[\inf_\jmath a_\jmath, \sup_\jmath b_\jmath] \subset \mathcal{V}(\psi)$, for otherwise there would exist an $x \in \mathcal{L}_1$ such that $\inf_\jmath a_\jmath \prec x \prec \sup_\jmath b_\jmath$ and $x \notin \mathcal{V}(\psi)$; but this would imply the absurdity that either $\inf_\jmath a_\jmath \prec x \prec a_\jmath$ or $b_\jmath \prec x \prec \sup_\jmath b_\jmath$ holds for all $\jmath \in \mathcal{C}$. Thus it follows that $[\inf_\jmath a_\jmath, \sup_\jmath b_\jmath]$ is a closed-interval subset of $\mathcal{V}(\psi)$ and is therefore an element of $\mathcal{B}^\star$ that is also an upper bound of $\mathcal{C}$. This completes the proof.

Corollary 14 *If $\psi \in \mathcal{O}_{12}^G$ and if the kernel of ψ is closed under monotone limits, then $\mathcal{B}_\star$ exists and is equal to $\bigcap \mathbf{B}(\psi)$.*

I now introduce several concepts whose development will make it possible **to explicitly determine the maps in Λ_{12}^G, $\mathcal{E}_{12}^G$, and $\widetilde{\mathcal{D}}_{12}^G$**, which in turn will lead to **an explicit determination of the maps in $\mathcal{D}_{21}^G$** and $\widetilde{\mathcal{D}}_{21}^G$.

3.4.2 Kernel Sets

The mapping $\psi \longmapsto \mathcal{V}(\psi)$ of $\mathcal{O}_{12}^G$ into $\mathcal{P}(\mathcal{L}_1)$ turns out to be a lattice isomorphism onto the collection of sets $\{\mathcal{V}(\psi) : \psi \in \mathcal{O}_{12}^G\}$, where the meet and join operations for subsets of $\mathcal{L}_1$ are assumed to be $\cap$ and $\cup$, respectively. This is shown, among other things, in the course of the development I detail in this section.

There is a many-to-one mapping of the subsets of $\mathcal{L}_1$ onto $\mathcal{O}_{12}^G$ that is closely related to the correspondence between the maps in $\mathcal{O}_{12}^G$ and their kernels.

Definition 33 *If $\mathcal{A} \subset \mathcal{L}_1$, then* **define the mapping** *$\psi_\mathcal{A} \in \mathcal{O}_{12}$ by*

$$\psi_\mathcal{A}(x) = \bigvee\{\eta \in \ell : \theta_{-\eta}(x) \in \mathcal{A}\}.$$

Proposition 27 *If $\mathcal{A} \subset \mathcal{L}_1$, then $\psi_\mathcal{A} \in \mathcal{O}_{12}^G$ and $\mathcal{V}(\psi_\mathcal{A}) \supset \mathcal{A}$.*

Proof First we show that $\tau_\xi(\psi_\mathcal{A}(x)) = \psi_\mathcal{A}(\theta_\xi(x))$ for all $x \in \mathcal{L}_1$ and all $\xi \in \ell$. We begin with $\tau_\xi(\psi_\mathcal{A}(x)) = \bigvee\{\xi + \eta : \eta \in \ell, \theta_{-\eta}(x) \in \mathcal{A}\}$. Putting $\xi + \eta = \beta$, we then get

$$\bigvee\{\xi + \eta : \eta \in \ell, \theta_{-\eta}(x) \in \mathcal{A}\} = \bigvee\{\beta \in \ell : \theta_{\xi-\beta}(x) \in \mathcal{A}\} = \psi_\mathcal{A}(\theta_\xi(x)).$$

Now, $\mathcal{V}(\psi_\mathcal{A}) = \{a \in \mathcal{L}_1 : \rho \preceq \psi_\mathcal{A}(a)\}$, by definition. Moreover, $\rho \preceq \psi_\mathcal{A}(a)$ is equivalent to $\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{A}\}$. If $b \in \mathcal{A}$, then $\rho \in \{\eta \in \ell : \theta_{-\eta}(b) \in \mathcal{A}\}$, and it follows that $\rho \preceq \psi_\mathcal{A}(b)$. Hence $\mathcal{V}(\psi_\mathcal{A}) \supset \mathcal{A}$.

Thus $\mathcal{A} \longmapsto \psi_{\mathcal{A}}$ is a mapping of $\mathcal{P}(\mathcal{L}_1)$ into $\mathcal{O}_{12}^G$.

Definition 34 *If $\mathcal{V} \subset \mathcal{L}_1$ satisfies $a \in \mathcal{V} \iff \rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{V}\}$, then $\mathcal{V}$ will be called a* **kernel set**. *We* **denote the collection of all kernel sets** *by $\mathcal{K}_1$.*

Remark 19 *A subset $\mathcal{A}$ of $\mathcal{L}_1$ is a kernel set if and only if*

$$\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{A}\} \Longrightarrow a \in \mathcal{A}.$$

Proposition 28 *If $\mathcal{V} \subset \mathcal{L}_1$ is a kernel set, then $\theta_\xi(\mathcal{V})$ is a kernel set for all $\xi \in \ell$; in fact, if α is any automorphism of $\mathcal{L}_1$ that commutes with θ_η for all $\eta \in \ell$, then $\alpha(\mathcal{V}) \in \mathcal{K}_1$.*

Proof For each $\xi \in \ell$, it is clear that θ_ξ is an automorphism of $\mathcal{L}_1$ that commutes with θ_η for all $\eta \in \ell$; hence, it is sufficient to prove the last part of the proposition, namely: Given that $\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{V}\} \Longrightarrow a \in \mathcal{V}$, show that

$$\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \alpha(\mathcal{V})\} \Longrightarrow a \in \alpha(\mathcal{V}).$$

For this we note that $\theta_{-\eta}(a) \in \alpha(\mathcal{V})$ asserts precisely that $\theta_{-\eta}(a) = \alpha(b)$ for some $b \in \mathcal{V}$. Since α is an automorphism, we therefore have $\theta_{-\eta}(a) \in \alpha(\mathcal{V}) \iff \alpha^{-1} \circ \theta_{-\eta}(a) \in \mathcal{V}$. By the commutativity assumed, then, $\theta_{-\eta}(a) \in \alpha(\mathcal{V}) \iff \theta_{-\eta}(\alpha^{-1}(a)) \in \mathcal{V}$. Thus $\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \alpha(\mathcal{V})\} \iff \rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(\alpha^{-1}(a)) \in \mathcal{V}\}$, which implies that $\alpha^{-1}(a) \in \mathcal{V}$ and hence that $a \in \alpha(\mathcal{V})$.

Lemma 11 *Let $\mathcal{A} \subset \mathcal{L}_1$ and let $\psi \in \mathcal{O}_{12}^G$. Then we have the following.*

1. *$\mathcal{V}(\psi_{\mathcal{A}}) = \mathcal{A}$ if and only if $\mathcal{A}$ is a kernel set.*
2. *$\mathcal{V}(\psi)$ is a kernel set.*

Proof (1) Suppose that $\mathcal{A} \in \mathcal{K}_1$. Since $\mathcal{V}(\psi_{\mathcal{A}}) = \{a \in \mathcal{L}_1 : \rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{A}\}$, and

$$\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{A}\} \Longrightarrow a \in \mathcal{A}$$

(since $\mathcal{A}$ is a kernel set), we get $\mathcal{V}(\psi_{\mathcal{A}}) \subset \mathcal{A}$. Hence, $\mathcal{V}(\psi_{\mathcal{A}}) = \mathcal{A}$.

Now suppose that $\mathcal{V}(\psi_{\mathcal{A}}) = \mathcal{A}$. Then we have that

$$\mathcal{A} = \{a \in \mathcal{L}_1 : \rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{A}\}\},$$

i.e., every $a \in \mathcal{L}_1$ that satisfies $\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{A}\}$ must lie in $\mathcal{A}$.

(2) Put $\mathcal{V}(\psi) = \mathcal{A}$. Since $\psi = \psi_{\mathcal{A}} \Longrightarrow \mathcal{A} = \mathcal{V}(\psi) = \mathcal{V}(\psi_{\mathcal{A}})$, and since the latter is equivalent to $\mathcal{A} = \mathcal{V}(\psi)$ is a kernel set, it is enough to prove that $\psi = \psi_{\mathcal{A}}$. For each $x \in \mathcal{L}_1$, we have, on the one hand, that $\psi_{\mathcal{A}}(x) = \bigvee\{\eta \in \ell : \theta_{-\eta}(x) \in \mathcal{A}\}$ (by definition), and on the other that $\psi(x) = \bigvee_{[a,b]\subset\mathcal{A}} \bigvee\{\eta \in \ell : \theta_\eta(a) \preceq x \preceq \theta_\eta(b)\}$ (by Ronse's theorem). Now, it is clear that

$$\{\eta \in \ell : \theta_\eta(a) \preceq x \preceq \theta_\eta(b)\} = \{\eta \in \ell : a \preceq \theta_{-\eta}(x) \preceq b\},$$

so that $\psi(x) = \bigvee\{\eta \in \ell : \theta_{-\eta}(x) \in [a,b] \subset \mathcal{A}\}$. Moreover, every $\eta \in \ell$ that satisfies $\theta_{-\eta}(x) \in \mathcal{A}$ also satisfies $\theta_{-\eta}(x) \in [\theta_{-\eta}(x), \theta_{-\eta}(x)] \subset \mathcal{A}$; conversely, every $\eta \in \ell$ that satisfies $\theta_{-\eta}(x) \in [a,b] \subset \mathcal{A}$ also satisfies $\theta_{-\eta}(x) \in \mathcal{A}$. This completes the proof.

Corollary 15 *We now have the following two results.*

1. $\mathcal{A} \longmapsto \psi_{\mathcal{A}}$ *maps* $\mathcal{P}(\mathcal{L}_1)$ *onto* $\mathcal{O}_{12}^G$.
2. $\psi \longmapsto \mathcal{V}(\psi)$ *is a bijection of* $\mathcal{O}_{12}^G$ *onto* $\mathcal{K}_1$.

Proposition 29 $\mathcal{K}_1$ *is a complete lattice relative to* $\bigcap$ *and* $\bigcup$.

Proof Let $\{\mathcal{V}_\alpha\}$ be any collection of kernel sets. Then for all α we have

$$\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \mathcal{V}_\alpha\} \Longrightarrow a \in \mathcal{V}_\alpha.$$

Thus

$$\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \bigcap_\alpha \mathcal{V}_\alpha\} \Longrightarrow a \in \bigcap_\alpha \mathcal{V}_\alpha,$$

and

$$\rho \preceq \bigvee\{\eta \in \ell : \theta_{-\eta}(a) \in \bigcup_\alpha \mathcal{V}_\alpha\} \Longrightarrow a \in \bigcup_\alpha \mathcal{V}_\alpha.$$

Corollary 16 *If* $\{\mathcal{V}_\alpha\}$ *is any collection of kernel sets, then*

$$\inf\{\mathcal{V}_\alpha\} = \bigcap_\alpha \mathcal{V}_\alpha \quad \text{and} \quad \sup\{\mathcal{V}_\alpha\} = \bigcup_\alpha \mathcal{V}_\alpha.$$

Remark 20 $\mathcal{V}(\mathbf{o}) = \emptyset$ *and* $\mathcal{V}(\mathbf{e}) = \mathcal{L}_1$*; hence,* $\inf \mathcal{K}_1 = \emptyset$ *and* $\sup \mathcal{K}_1 = \mathcal{L}_1$.

Theorem 6 $\psi \longmapsto \mathcal{V}(\psi)$ *is a lattice isomorphism of* $\mathcal{O}_{12}^G$ *onto* $\mathcal{K}_1$.

Proof Let $\psi, \phi \in \mathcal{O}_{12}^G$. Then for all $x \in \mathcal{L}_1$ we have

$$(\psi \wedge \phi)(x) = \psi(x) \wedge \phi(x) \quad \text{and} \quad (\psi \vee \phi)(x) = \psi(x) \vee \phi(x).$$

Thus it follows that $\mathcal{V}(\psi \wedge \phi) = \{x \in \mathcal{L}_1 : \rho \preceq \psi(x) \wedge \phi(x)\} =$

$$\{x \in \mathcal{L}_1 : \rho \preceq \psi(x)\} \cap \{x \in \mathcal{L}_1 : \rho \preceq \phi(x)\} = \mathcal{V}(\psi) \cap \mathcal{V}(\phi).$$

In the same way, $\mathcal{V}(\psi \vee \phi) = \{x \in \mathcal{L}_1 : \rho \preceq \psi(x) \vee \phi(x)\} =$

$$\{x \in \mathcal{L} : \rho \preceq \psi(x)\} \cup \{x \in \mathcal{L}_1 : \rho \preceq \phi(x)\} = \mathcal{V}(\psi) \cup \mathcal{V}(\phi).$$

Definition 35 *If* $\mathcal{A} \subset \mathcal{L}_1$, *then*

$$\mathcal{A}\| \equiv \bigcap\{\mathcal{V} \in \mathcal{K}_1 : \mathcal{A} \subset \mathcal{V}\}$$

defines the **kernel closure of** $\mathcal{A}$.

Remark 21 *Since the intersection of any collection of kernel sets is a kernel set, it follows that* $\mathcal{A}\|$ *is the smallest kernel set containing* $\mathcal{A}$.

Proposition 30 *If $\mathcal{A} \subset \mathcal{L}_1$, then $\mathcal{V}(\psi_{\mathcal{A}}) = \mathcal{A}\|$.*

Proof Since $\psi_{\mathcal{A}} \in \mathcal{O}_{12}^G$, even when $\mathcal{A}$ is not a kernel set, it follows that there is a unique $\mathcal{V} \in \mathcal{K}_1$ such that $\mathcal{V}(\psi_{\mathcal{A}}) = \mathcal{V}$. This proposition asserts that $\mathcal{V}$ is $\mathcal{A}\|$. Since the kernel of $\psi_{\mathcal{A}}$ contains $\mathcal{A}$, it is clear that $\mathcal{V} \supset \mathcal{A}\|$. On the other hand, since $\mathcal{A} \subset \mathcal{A}\|$, it follows from the definition of $\psi_{\mathcal{A}}$ that $\psi_{\mathcal{A}} \preceq \psi_{\mathcal{A}\|}$. Because $\psi \longmapsto \mathcal{V}(\psi)$ is a lattice isomorphism, it finally follows that $\mathcal{V} = \mathcal{V}(\psi_{\mathcal{A}}) \subset \mathcal{V}(\psi_{\mathcal{A}\|}) = \mathcal{A}\|$. This completes the proof.

Corollary 17 *The relation $\sim$ defined in $\mathcal{P}(\mathcal{L}_1)$ by $\mathcal{A} \sim \mathcal{B} \iff \mathcal{A}\| = \mathcal{B}\|$ is an equivalence relation that consequently partitions $\mathcal{P}(\mathcal{L}_1)$. There is thus a bijection $\psi \longmapsto \mathbf{A}_\psi$ of $\mathcal{O}_{12}^G$ onto the set of $\sim$-equivalence classes such that $\mathcal{A} \in \mathbf{A}_\psi \iff \psi_{\mathcal{A}} = \psi$. Hence, for each $\psi \in \mathcal{O}_{12}^G$ it follows that $\bigcup \mathbf{A}_\psi = \mathcal{V}(\psi)$ and $\bigcap \mathbf{A}_\psi \equiv \widehat{\mathcal{A}}_\psi$ is the least element of $\mathcal{P}(\mathcal{L}_1)$ such that $\psi_{\widehat{\mathcal{A}}_\psi} = \psi$.*

We are now in a position **to explicitly determine the maps in $\mathbf{\Lambda}_{12}^G$, $\mathcal{E}_{12}^G$, and $\widetilde{\mathcal{D}}_{12}^G$.** This will in turn lead to **an explicit determination of the maps in $\mathcal{D}_{21}^G$** and $\widetilde{\mathcal{D}}_{21}^G$.

3.4.3 Sup-Generating G-Invariant Mappings

Lemma 12 *If $[a,b]$ is a closed interval of $\mathcal{L}_1$, then $\bigcap\{\theta_\eta([a,b]) : \eta \in \ell(y)\}$ is either empty or a closed interval for all $y \in \mathcal{L}_2$; recall that $\ell(y) = \{\eta \in \ell : \eta \preceq y\}$.*

Proof Since it is clear that $\theta_\eta([a,b])$ is a closed interval of $\mathcal{L}_1$ for all $\eta \in \ell$, we will prove that any intersection of closed intervals is either empty or a closed interval. Let $\{[a_\beta . b_\beta]\}$ be an arbitrary collection of closed intervals of $\mathcal{L}_1$. If $\bigcap_\beta [a_\beta, b_\beta] = \emptyset$, we have nothing to prove. If $\bigcap_\beta [a_\beta, b_\beta] = \mathcal{A} \neq \emptyset$, then let x be an element of $\mathcal{A}$. Thus $a_\beta \preceq x \preceq b_\beta$ for all β and we see that $\sup_\beta a_\beta \preceq x \preceq \inf_\beta b_\beta$. Hence $\mathcal{A} \subset [\sup_\beta a_\beta, \inf_\beta b_\beta]$. Now suppose that x is an element of $\mathcal{L}_1$ that satisfies $\sup_\beta a_\beta \preceq x \preceq \inf_\beta b_\beta$. Then $a_\beta \preceq x \preceq b_\beta$ for all β and we see that $x \in \mathcal{A}$, which inturn implies that $\mathcal{A} \supset [\sup_\beta a_\beta, \inf_\beta b_\beta]$.

Lemma 13 *If $a, b \in \mathcal{L}_1$, then we have the following.*

1. *If $\psi \in \mathcal{O}_{12}^G$ and $\mathcal{V}(\psi) = [a,b]$, then $\psi = \sigma_{ab}$.*
2. *$\mathcal{V}(\sigma_{ab}) = [a,b]\|$.*

Proof (1) By the corollary to Ronse's theorem (Cor. 13),

$$\psi = \sup_{\mathcal{O}_{12}^G}\{\sigma_{cd} : [c,d] \subset [a,b]\}.$$

If $[c,d] \subset [a,b]$, then for all $x \in \mathcal{L}_1$, we have

$$\sigma_{cd}(x) = \sup\{\eta \in \ell : \theta_{-\eta}(x) \in [c,d]\} \preceq \sup\{\eta \in \ell : \theta_{-\eta}(x) \in [a,b]\} = \sigma_{ab}(x).$$

Thus $\sigma_{cd} \preceq_p \sigma_{ab}$, and it follows that $\psi = \sigma_{ab}$.

(2) First we note that $\psi_{[a,b]}(x) = \sigma_{ab}(x)$ for all $x \in \mathcal{L}_1$; indeed, by definition we have

$$\psi_{[a,b]}(x) = \bigvee\{\eta \in \ell : a \preceq \theta_{-\eta}(x) \preceq b\},$$

and, since

$$a \preceq \theta_{-\eta}(x) \preceq b \iff \theta_\eta(a) \preceq x \preceq \theta_\eta(b),$$

we also have

$$\bigvee\{\eta \in \ell : a \preceq \theta_{-\eta}(x) \preceq b\} = \bigvee\{\eta \in \ell : \theta_\eta(a) \preceq x \preceq \theta_\eta(b)\} = \sigma_{ab}(x).$$

This shows that $\mathcal{V}(\sigma_{ab}) = [a,b]||$.

Theorem 7 $\mathbf{\Lambda}^G_{12} = \{\sigma_{ab} : [a,b] \in \mathcal{K}_1\}$.

Proof If $\psi \in \mathbf{\Lambda}^G_{12}$ and $\psi \neq \mathbf{o}$, then $\cdot\mathcal{K}(\psi)(\rho) = [a,b]$ for some closed interval $[a,b]$. Hence $[a,b] = \mathcal{V}(\psi)$, is therefore a kernel set, and $\psi = \sigma_{ab}$ by the last lemma. If $\psi = \mathbf{o}$, then choose any $a, b \in \mathcal{L}_1$ such that $a \not\preceq b$.

On the other hand, suppose that $[a,b] \in \mathcal{K}_1$. We complete the proof by showing that σ_{ab} is sup-generating. Since $\mathcal{V}(\sigma_{ab}) = [a,b]||$ and $[a,b] \in \mathcal{K}_1$, it follows that $\mathcal{V}(\sigma_{ab}) = [a,b]$. If $[a,b] \neq \emptyset$, then for all $y \in \mathcal{L}_2$ we have

$$\cdot\mathcal{K}(\sigma_{ab})(y) = \bigcap\{\theta_\eta([a,b]) : \eta \in \ell(y)\},$$

which is either empty or a closed interval. If $[a,b] = \emptyset$, then $\cdot\mathcal{K}(\sigma_{ab})(y) = \emptyset$ for all $y \in \mathcal{L}_2$. Thus we have proved that σ_{ab} is sup-generating.

We now compare, in light of this theorem, **the Ronse decomposition of a G-invariant map, ψ, with what would be the G-invariant analog of (a) of Theorem 2**; that is, we compare $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \subset \mathcal{V}(\psi)\}$ with $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \in \mathcal{K}_1, \sigma_{ab} \preceq_p \psi\}$.

Theorem 8 *If all closed intervals of $\mathcal{L}_1$ are kernel sets, then $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \subset \mathcal{V}(\psi)\} = \sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \in \mathcal{K}_1, \sigma_{ab} \preceq_p \psi\}$. If not all closed intervals of $\mathcal{L}_1$ are kernel sets, then it can happen that there are σ_{ab} that are not sup-generating, which are therefore not present in the formula $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \in \mathcal{K}_1, \sigma_{ab} \preceq_p \psi\}$, but which are nonetheless still needed for the Ronse decomposition.*

Proof For the first part, rewrite the Ronse decomposition as $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : \mathcal{V}(\sigma_{ab}) \subset \mathcal{V}(\psi)\} = \sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : \sigma_{ab} \preceq_p \psi\}$, which equals $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \in \mathcal{K}_1, \sigma_{ab} \preceq_p \psi\}$, since all $[a,b] \in \mathcal{K}_1$. Section 4.1 provides the example that proves the second part.

In situations where a smallest basis $\mathcal{B}_\star$ for ψ exists, we of course have the most economical Ronse decomposition $\sup_{\mathcal{O}^G_{12}}\{\sigma_{ab} : [a,b] \in \mathcal{B}_\star\}$. **But even in this case it is not clear that the maximal closed intervals of $\mathcal{B}_\star$ are necessarily kernel sets.**

3.4.4 G-Invariant Erosions and Anti-Dilations

Lemma 14 *If $\varepsilon \in \mathcal{E}_{12}^G$ and $\widetilde{\delta} \in \widetilde{\mathcal{D}}_{12}^G$, then we have the following.*

1. $\mathcal{V}(\varepsilon) = [a, E]$ *for some* $a \in \mathcal{L}_1$.
2. $\mathcal{V}(\widetilde{\delta}) = [O, b]$ *for some* $b \in \mathcal{L}_1$.

Proof (1) By definition, $\mathcal{V}(\varepsilon) = \{x \in \mathcal{L}_1 : \rho \preceq \varepsilon(x)\}$. Since $\varepsilon(\inf \mathcal{B}) = \inf\{\varepsilon(x) : x \in \mathcal{B}\}$ for all $\mathcal{B} \subset \mathcal{L}_1$, it follows in particular that $\varepsilon(\inf \mathcal{V}(\varepsilon)) = \inf\{\varepsilon(x) : x \in \mathcal{V}(\varepsilon)\}$. Thus we see that $\rho \preceq \inf\{\varepsilon(x) : x \in \mathcal{V}(\varepsilon)\} = \varepsilon(\inf \mathcal{V}(\varepsilon))$, i.e., $\inf \mathcal{V}(\varepsilon) \in \mathcal{V}(\varepsilon)$. Let $a = \inf \mathcal{V}(\varepsilon)$. It follows that every $x \in \mathcal{V}(\varepsilon)$ satisfies $a \preceq x$. Furthermore, if x is any element of $\mathcal{L}_1$ such that $a \prec x$, then, since erosions are increasing, it follows that $\rho \preceq \varepsilon(a) \preceq \varepsilon(x)$. Hence, we have shown that $\mathcal{V}(\varepsilon) = [a, E]$.

(2) By definition, $\mathcal{V}(\widetilde{\delta}) = \{x \in \mathcal{L}_1 : \rho \preceq \widetilde{\delta}(x)\}$. Since $\widetilde{\delta}(\sup \mathcal{B}) = \inf\{\widetilde{\delta}(x) : x \in \mathcal{B}\}$, for all $\mathcal{B} \subset \mathcal{L}_1$, it follows in particular that $\widetilde{\delta}(\sup \mathcal{V}(\widetilde{\delta})) = \inf\{\widetilde{\delta}(x) : x \in \mathcal{V}(\widetilde{\delta})\}$. Thus we see that $\rho \preceq \inf\{\widetilde{\delta}(x) : x \in \mathcal{V}(\widetilde{\delta})\} = \widetilde{\delta}(\sup \mathcal{V}(\widetilde{\delta}))$, i.e., $\sup \mathcal{V}(\widetilde{\delta}) \in \mathcal{V}(\widetilde{\delta})$. Let $b = \sup \mathcal{V}(\widetilde{\delta})$. It follows that every $x \in \mathcal{V}(\widetilde{\delta})$ satisfies $x \preceq b$. Furthermore, if x is any element of $\mathcal{L}_1$ such that $x \prec b$, then, since anti-dilations are decreasing, it follows that $\rho \preceq \widetilde{\delta}(b) \preceq \widetilde{\delta}(x)$. Hence we have shown that $\mathcal{V}(\widetilde{\delta}) = [O, b]$.

Definition 36 *For each $a \in \mathcal{L}_1$* **define the mappings** $\varepsilon_a = \sigma_{aE}$ *and* $\widetilde{\delta}_a = \sigma_{Oa}$.

Lemma 15 *Let $a, b \in \mathcal{L}_1$. Then $\varepsilon_a \in \mathcal{O}_{12}^{G+}$, $\widetilde{\delta}_b \in \mathcal{O}_{12}^{G-}$, and we also have the following.*

1. $\varepsilon_a(x) = \sup\{\eta \in \ell : \theta_\eta(a) \preceq x\}$ *for all* $x \in \mathcal{L}_1$.
2. $\widetilde{\delta}_a(x) = \sup\{\eta \in \ell : x \preceq \theta_\eta(a)\}$ *for all* $x \in \mathcal{L}_1$.
3. *If $[a, E] \in \mathcal{K}_1$, then ε_a is a G-invariant erosion whose kernel is $[a, E]$.*
4. *If $[O, b] \in \mathcal{K}_1$, then $\widetilde{\delta}_b$ is a G-invariant anti-dilation whose kernel is $[O, b]$.*

Proof (3) It is clear that $\varepsilon_a = \psi_{[a,E]}$ is G-invariant and that $\mathcal{V}(\varepsilon_a) = [a, E]$. We must show for any $\mathcal{B} \subset \mathcal{L}_1$ that $\varepsilon_a(\inf \mathcal{B}) = \inf \varepsilon_a(\mathcal{B})$. Since ε_a is increasing, it is clear that $\varepsilon_a(\inf \mathcal{B}) \preceq \inf \varepsilon_a(\mathcal{B})$. Since $[a, E] \in \mathcal{K}_1$, it follows that $\sigma_{aE} = \varepsilon_a$ is sup-generating, i.e.,

$$\varepsilon_a(\inf \mathcal{B}) \wedge \varepsilon_a(\sup \mathcal{B}) = \inf \varepsilon_a(\mathcal{B}).$$

Hence, $\inf \varepsilon_a(\mathcal{B}) \preceq \varepsilon_a(\inf \mathcal{B})$.

(4) It is clear that $\widetilde{\delta}_b = \psi_{[O,b]}$ is G-invariant and that $\mathcal{V}(\widetilde{\delta}_b) = [O, b]$. We must show for any $\mathcal{B} \subset \mathcal{L}_1$ that $\widetilde{\delta}_b(\sup \mathcal{B}) = \inf \widetilde{\delta}_b(\mathcal{B})$. Since $\widetilde{\delta}_b$ is decreasing, it is clear that $\widetilde{\delta}_b(\sup \mathcal{B}) \preceq \inf \widetilde{\delta}_b(\mathcal{B})$. Since $[O, b] \in \mathcal{K}_1$, it follows that $\sigma_{Ob} = \widetilde{\delta}_b$ is sup-generating, i.e.,

$$\widetilde{\delta}_b(\inf \mathcal{B}) \wedge \widetilde{\delta}_b(\sup \mathcal{B}) = \inf \widetilde{\delta}_b(\mathcal{B}).$$

Hence, $\inf \widetilde{\delta}_b(\mathcal{B}) \preceq \widetilde{\delta}_b(\sup \mathcal{B})$.

Theorem 9 *Denote the sets* $\{a \in \mathcal{L}_1 : [a, E] \in \mathcal{K}_1\}$ *and* $\{b \in \mathcal{L}_1 : [O, b] \in \mathcal{K}_1\}$ *by* $[\mathcal{K}_1, E]$ *and* $[O, \mathcal{K}_1]$*, respectively. Then we have the following.*

1. $\mathcal{E}_{12}^G = \{\varepsilon_a : a \in [\mathcal{K}_1, E]\}$.
2. $\widetilde{\mathcal{D}}_{12}^G = \{\widetilde{\delta}_b : b \in [O, \mathcal{K}_1]\}$.
3. $[\mathcal{K}_1, E]$ *is a join-complete sublattice of* $\mathcal{L}_1$.
4. $[O, \mathcal{K}_1]$ *is a meet-complete sublattice of* $\mathcal{L}_1$.
5. $a \longmapsto [a, E]$ *is a dual-lattice isomorphism of* $[\mathcal{K}_1, E]$ *onto* $(\{[a, E] : a \in [\mathcal{K}_1, E]\}, \cap, \cup)$.
6. $b \longmapsto [O, b]$ *is a lattice isomorphism of* $[O, \mathcal{K}_1]$ *onto* $(\{[O, b] : b \in [O, \mathcal{K}_1]\}, \cap, \cup)$.
7. $\varepsilon_a \longmapsto a$ *is a dual-lattice isomorphism of* $\mathcal{E}_{12}^G$ *onto* $[\mathcal{K}_1, E]$.
8. $\widetilde{\delta}_b \longmapsto b$ *is a lattice isomorphism of* $\widetilde{\mathcal{D}}_{12}^G$ *onto* $[O, \mathcal{K}_1]$.

Proof (1) and (2) are immediate consequences of the last two lemmas. The identities (A) and (B) below show that $[\mathcal{K}_1, E]$ and $[O, \mathcal{K}_1]$ are sublattices of $\mathcal{L}_1$.

(A) $[a_1, E] \cap [a_2, E] = [a_1 \vee a_2, E]$ and $[a_1, E] \cup [a_2, E] = [a_1 \wedge a_2, E]$ for all $a_1, a_2 \in \mathcal{L}_1$

(B) $[O, b_1] \cap [O, b_2] = [O, b_1 \wedge b_2]$ and $[O, b_1] \cup [O, b_2] = [O, b_1 \vee b_2]$ for all $b_1, b_2 \in \mathcal{L}_1$

These identities also furnish easy proofs of (5), (6), (7), and (8). Since $O \in [\mathcal{K}_1, E]$ and since $A \subset [\mathcal{K}_1, E] \Longrightarrow \sup A \in [\mathcal{K}_1, E]$, it follows that $[\mathcal{K}_1, E]$ is a join-complete sublattice of $\mathcal{L}_1$. Likewise, since $E \in [O, \mathcal{K}_1]$ and since

$$B \subset [O, \mathcal{K}_1] \Longrightarrow \inf B \in [O, \mathcal{K}_1],$$

it follows that $[O, \mathcal{K}_1]$ is a meet-complete sublattice of $\mathcal{L}_1$.

Definition 37 *For each* $a \in [\mathcal{K}_1, E]$ *and* $b \in [O, \mathcal{K}_1]$ **define the maps** ς_a *and* $\widetilde{\varsigma}_b$ *as follows.*

1. ς_a *is the unique dilation in* $\mathcal{D}_{21}^G$ *such that* $(\varepsilon_a, \varsigma_a)$ *is a G-adjunction.*
2. $\widetilde{\varsigma}_b$ *is the unique anti-dilation in* $\widetilde{\mathcal{D}}_{21}^G$ *such that* $(\widetilde{\delta}_b, \widetilde{\varsigma}_b)$ *is a G-invariant Galois connection.*

Remark 22 *With regard to* ς_a *and* $\widetilde{\varsigma}_b$ *we have the following.*

1. $\varepsilon_a \longmapsto \varsigma_a$ *is a dual-lattice isomorphism of* $\mathcal{E}_{12}^G$ *onto* $\mathcal{D}_{21}^G$.
2. $\widetilde{\delta}_b \longmapsto \widetilde{\varsigma}_b$ *is a lattice isomorphism of* $\widetilde{\mathcal{D}}_{12}^G$ *onto* $\widetilde{\mathcal{D}}_{21}^G$.
3. $\varsigma_a \longmapsto a$ *is a lattice isomorphism of* $\mathcal{D}_{21}^G$ *onto* $[\mathcal{K}_1, E]$.
4. $\widetilde{\varsigma}_b \longmapsto b$ *is a lattice isomorphism of* $\widetilde{\mathcal{D}}_{21}^G$ *onto* $[O, \mathcal{K}_1]$.

Lemma 16 *If $a \in [\mathcal{K}_1, E]$ and $b \in \mathcal{L}_1$, then we have that*

1. $[a,b]\| \subset [a,E] \cap [O,b]\|$ *(equivalently, $\sigma_{ab} \preceq_p \varepsilon_a \wedge \widetilde{\delta}_b$).*
2. *If $b \in [O, \mathcal{K}_1]$, then $[a,b]$ is a kernel set and $\sigma_{ab} = \varepsilon_a \wedge \widetilde{\delta}_b$.*

Proof (1) Since $[a,E]$ and $[O,b]\|$ are kernel sets, it follows that $[a,E] \cap [O,b]\|$ is a kernel set; moreover, $[a,b] = [a,E] \cap [O,b] \subset [a,E] \cap [O,b]\|$.

(2) Since $[a,E]$ and $[O,b]$ are now kernel sets, it follows that $[a,E] \cap [O,b] = [a,b]$ is a kernel set, and hence that $\sigma_{ab} = \varepsilon_a \wedge \widetilde{\delta}_b$.

The next result is a corollary of Ronse's theorem.

Corollary 18 *If $\psi \in \mathcal{O}_{12}^{G+}$ and $\phi \in \mathcal{O}_{12}^{G-}$, then we have that*

1. $\psi(x) = \bigvee\{\varepsilon_a(x) : a \in \mathcal{V}(\psi)\}$ *for all $x \in \mathcal{L}_1$.*
2. $\phi(x) = \bigvee\{\widetilde{\delta}_b(x) : b \in \mathcal{V}(\phi)\}$ *for all $x \in \mathcal{L}_1$.*

Proof (1) We apply the Ronse decomposition,

$$\psi(x) = \bigvee\{\sigma_{ab}(x) : (a,b) \in \mathcal{W}(\psi)\},$$

to the case where ψ is increasing. In this case we have $a \in \mathcal{V}(\psi) \Longrightarrow (a,E) \in \mathcal{W}(\psi)$. Indeed, since $a \in \mathcal{V}(\psi) \iff \rho \preceq \psi(a)$, and ψ is increasing, it is clear that

$$a \preceq x' \preceq E \Longrightarrow \rho \preceq \psi(a) \preceq \psi(x').$$

Moreover, if $(a,b) \in \mathcal{W}(\psi)$ and $x \in \mathcal{L}_1$, then $\theta_\eta(a) \preceq x \preceq \theta_\eta(b) \Longrightarrow \theta_\eta(a) \preceq x$, and so we have that $\sigma_{ab}(x) \preceq \sigma_{aE}(x) = \varepsilon_a(x)$. Therefore

$$\psi(x) = \bigvee\{\sigma_{aE}(x) : a \in \mathcal{V}(\psi)\} = \bigvee\{\varepsilon_a(x) : a \in \mathcal{V}(\psi)\}.$$

(2) A proof much like that of (1).

We should finally note that **the theory developed so far has nothing to say about** the constitutions of $\mathcal{E}_{21}^G$ and $\mathcal{D}_{12}^G$, nor does it tell us anything about the constitutions of $\mathbf{M}_{12}^G$, $\widetilde{\mathcal{E}}_{12}^G$, and $\widetilde{\mathcal{E}}_{21}^G$. **To obtain such information, a theory of $\mathbf{M}_{12}^G$** that parallels what I have done for **$\mathbf{\Lambda}_{12}^G$ would have to be developed. Such a theory would have to be based on an inf-generating set** ℓ rather than a sup-generating one, **and would entail developing a theory of right kernels** analogous to the left-kernel theory developed here. The result of all this would be a theory of $\mathbf{M}_{12}^G$ that gives us explicit determinations of the maps in $\mathbf{M}_{12}^G$, $\mathcal{D}_{12}^G$, and $\widetilde{\mathcal{E}}_{12}^G$, just as the parallel theory of $\mathbf{\Lambda}_{12}^G$ has given us the constitutions of $\mathbf{\Lambda}_{12}^G$, $\mathcal{E}_{12}^G$, and $\widetilde{\mathcal{D}}_{12}^G$. Having the constitution of $\mathbf{M}_{12}^G$, moreover, we could then simply employ the lattice isomorphism $\mu \longmapsto (\underline{\mu\cdot}, \overline{\mu\cdot})$ to generate the $\underline{\mu\cdot} \in \widetilde{\mathcal{E}}_{21}^G$ and $\overline{\mu\cdot} \in \mathcal{E}_{21}^G$ from the $\mu \in \mathbf{M}_{12}^G$.

3.4.5 G-Invariant Lattice Operators

The theory of sections 3.4.3 and 3.4.4 of course specializes to the case of G-invariant lattice operators, i.e., the case where $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}$ and $\mathcal{O}_{12} = \mathcal{O}_{21} = \mathcal{O}(\mathcal{L}) = \mathcal{O}$. This case has certain distinctive features that we will here elaborate. First **we adopt the following notations**: $\mathcal{E}_G$, $\mathcal{D}_G$, $\widetilde{\mathcal{E}}_G$, and $\widetilde{\mathcal{D}}_G$ denote, respectively, the sets of G-invariant erosions, dilations, anti-erosions, and anti-dilations on $\mathcal{L}$; $\mathcal{O}_G$, $\mathcal{O}_G^+$, and $\mathcal{O}_G^-$ respectively denote the G-invariant operators, G-invariant increasing operators, and G-invariant decreasing operators; finally, we let $\mathcal{K} = \mathcal{K}(\mathcal{L})$ denote the collection of all kernel sets.

Proposition 31 *$\mathcal{O}_G$, $\mathcal{O}_G^+$, $\mathcal{O}_G^-$, $\mathcal{E}_G$, $\mathcal{D}_G$, $\widetilde{\mathcal{E}}_G$, and $\widetilde{\mathcal{D}}_G$ each contain the identity operator,* $\mathbf{id} : x \longmapsto x$*, and are each closed under composition.*

For each $x \in \mathcal{L}$ and $\xi \in \ell$ **we introduce the handy notation**, $x_\xi \equiv \tau_\xi(x)$, of Heijmans and Ronse. The lemma below gives an alternative expression for the operator ε_a that closely resembles the well-known definition of *the erosion of a given set x by another set a.*

Lemma 17 $\varepsilon_a(x) = \bigwedge\{x_{-\xi} : \xi \in \ell(a)\}$ *for all $a, x \in \mathcal{L}$.*

Proof For all $a, x \in \mathcal{L}$ we show that $\bigvee\{\eta \in \ell : \tau_\eta(a) \preceq x\} = \bigwedge\{x_{-\xi} : \xi \in \ell(a)\}$. First we show that $\tau_\eta(a) \preceq x \Longrightarrow \eta \preceq x_{-\xi}$ $\forall\ \xi \in \ell$ such that $\xi \preceq a$. To see this, note that if $\xi \in \ell$, then $\tau_\eta(a) \preceq x$ implies that $\tau_{-\xi}\tau_\eta(a) \preceq \tau_{-\xi}(x)$, so that $\tau_{\eta-\xi}(a) \preceq \tau_{-\xi}(x)$. If $\xi \preceq a$, then $\rho = \tau_{-\xi}(\xi) \preceq \tau_{-\xi}(a)$. Therefore, $\eta = \tau_\eta(\rho) \preceq \tau_{\eta-\xi}(a) \preceq \tau_{-\xi}(x) = x_{-\xi}$. This proves the desired result, which in turn leads to the conclusion that

$$\bigvee\{\eta \in \ell : \tau_\eta(a) \preceq x\} \preceq \bigwedge\{x_{-\xi} : \xi \in \ell(a)\}.$$

To complete the proof we establish the reverse relation. For this, let us first notice that $\bigwedge\{x_{-\xi} : \xi \in \ell(a)\}$ is the supremum of the lower bounds of $\{x_{-\xi} : \xi \in \ell(a)\})$ that lie in ℓ (because ℓ is sup-generating). Accordingly, let $\zeta \in \ell$ satisfy $\zeta \preceq x_{-\xi}$ for all $\xi \in \ell(a)$. Then $\xi_\zeta = \xi + \zeta = \zeta_\xi \preceq \tau_\xi(x_{-\xi}) = x$. Since $a_\zeta = \bigvee\{\xi_\zeta : \xi \in \ell(a)\}$, it therefore follows that $a_\zeta \preceq x$; hence, $\zeta \preceq \bigvee\{\eta \in \ell : \tau_\eta(a) \preceq x\}$, and we see that

$$\bigwedge\{x_{-\xi} : \xi \in \ell(a)\} \preceq \bigvee\{\eta \in \ell : \tau_\eta(a) \preceq x\}.$$

Proposition 32 *The following four statements are equivalent and true for all $a \in \mathcal{L}$.*

1. *$[a, e]$ is the kernel of ε_a.*
2. *$[a, e] \in \mathcal{K}(\mathcal{L})$.*
3. *ε_a is a G-invariant erosion.*
4. *ε_a has a lower adjoint ς_a, i.e., $(\varepsilon_a, \varsigma_a)$ is an adjunction.*

Proof We prove (1). If $a \in \mathcal{L}$, then $\mathcal{V}(\varepsilon_a) = \{x \in \mathcal{L} : \rho \preceq \varepsilon_a(x)\}$. Moreover, $\rho \preceq \varepsilon_a(x)$ is equivalent to $\rho \preceq \bigwedge\{x_{-\xi} : \xi \in \ell(a)\}$, which implies that $\rho \preceq \tau_{-\xi}(x)$ for all $\xi \in \ell(a)$. Thus $\tau_\xi(\rho) \preceq \tau_\rho(x)$ for all $\xi \in \ell(a)$, which is the same as $\xi \preceq x$ for all $\xi \in \ell$ such that $\xi \preceq a$. Hence $a = \sup\{\xi \in \ell : \xi \preceq a\} \preceq x$, and this shows that $\mathcal{V}(\varepsilon_a) \subset [a, e]$.

We complete the proof by showing that $a \preceq x \Longrightarrow \rho \preceq \varepsilon_a(x)$. Now, $\sup \ell(a) = a \preceq x$; hence, $\xi \preceq x$ for all $\xi \in \ell$ such that $\xi \preceq a$, which is the same as $\tau_\xi(\rho) \preceq \tau_\rho(x)$ for all $\xi \in \ell(a)$. Thus $\rho \preceq \tau_{-\xi}(x)$ for all $\xi \in \ell(a)$, and it therefore follows that

$$\rho \preceq \bigwedge\{x_{-\xi} : \xi \in \ell(a)\} = \varepsilon_a(x).$$

From this result we obtain the following largely obvious corollary.

Corollary 19 *For all $a, x \in \mathcal{L}$ and $\xi \in \ell$, we have the following.*

1. $\varepsilon_a \in \mathcal{E}_G$, $\varsigma_a \in \mathcal{D}_G$, *and* $(\varepsilon_a, \varsigma_a)$ *is a G-adjunction.*
2. $\varsigma_a(x) = \bigvee\{x_\xi : \xi \in \ell(a)\}$.
3. $\varsigma_\xi = \tau_\xi$ *and* $\varepsilon_\xi = \tau_{-\xi}$.

Proof We will prove only (2). For this we show that

$$\bigwedge\{u \in \mathcal{L} : x \preceq \varepsilon_a(u)\} = \bigvee\{x_\xi : \xi \in \ell(a)\} \text{ for all } a, x \in \mathcal{L}.$$

We have $\bigwedge\{u \in \mathcal{L} : x \preceq \varepsilon_a(u)\} = \bigwedge\{u \in \mathcal{L} : x \preceq \bigwedge\{u_{-\xi} : \xi \in \ell(a)\}\} =$

$$\bigwedge\{u \in \mathcal{L} : x \preceq u_{-\xi}\ \forall\, \xi \in \ell(a)\} = \bigwedge\{u \in \mathcal{L} : x_\xi \preceq u\ \forall\, \xi \in \ell(a)\} \equiv \widehat{u}.$$

Since $\widehat{u}$ is the infimum of all the upper bounds u of the set $\{x_\xi : \xi \in \ell(a)\}$, it is clear that $\widehat{u}$ is simply the supremum $\bigvee\{x_\xi : \xi \in \ell(a)\}$ of that set. This completes the proof.

Note that the alternative expression in (2) above for the dilation operator ς_a closely resembles the well-known definition of *the dilation of a given set x by a structuring element a*, and in this regard is the natural companion of $\varepsilon_a(x) = \bigwedge\{x_{-\xi} : \xi \in \ell(a)\}$. It should also be noted that the isomorphisms given in Theorem 9 (6) and Remark 22 (3) are in fact onto $\mathcal{L}$ in the lattice operator case. In other words, we have that (1) $a \longmapsto \varepsilon_a$ is a dual-lattice isomorphism of $\mathcal{L}$ onto $\mathcal{E}_G$ and (2) $a \longmapsto \varsigma_a$ is a lattice isomorphism of $\mathcal{L}$ onto $\mathcal{D}_G$. Thus $a \longmapsto \varsigma_a$ preserves infima and suprema and $a \longmapsto \varepsilon_a$ reverses infima and suprema. The next remark spells this out in more detail.

Remark 23 *If $\mathcal{A} \subset \mathcal{L}$, then we have the following.*

1. $\varsigma_{\sup \mathcal{A}} = \sup\{\varsigma_\alpha : \alpha \in \mathcal{A}\}$ *and* $\varsigma_{\inf \mathcal{A}} = \inf\{\varsigma_\alpha : \alpha \in \mathcal{A}\}$.
2. $\varepsilon_{\sup \mathcal{A}} = \inf\{\varepsilon_\alpha : \alpha \in \mathcal{A}\}$ *and* $\varepsilon_{\inf \mathcal{A}} = \sup\{\varepsilon_\alpha : \alpha \in \mathcal{A}\}$.

Remark 24 *If $\mathcal{A} \subset \mathcal{L}$, $x \in \mathcal{L}$, and there exists an $\eta \in \ell$ such that $\rho \preceq \eta$ and $\tau_{-\eta}(x) \in \mathcal{A}$, then $x \in \mathcal{A}\| = \mathcal{V}(\psi_{\mathcal{A}})$; this is because $\mathcal{A} \subset \mathcal{A}\|$ and*

$$x \in \mathcal{A}\| \iff \rho \preceq \bigvee\{\eta \in \ell : \tau_{-\eta}(x) \in \mathcal{A}\|\}.$$

What can be said about this remark's converse? Indeed,

$$x \in \mathcal{A}\| \Longrightarrow \rho \preceq \vee\{\xi \in \ell : \tau_{-\xi}(x) \in \mathcal{A}\|\},$$

and we therefore have $x \in \mathcal{A}\| \Longrightarrow \rho \in \{\xi \in \ell : \tau_{-\xi}(x) \in \mathcal{A}\|\}$. But does $x \in \mathcal{A}\|$ imply the existence of an $\eta \in \ell$ such that $\tau_{-\eta}(x) \in \mathcal{A}$ and $\rho \preceq \eta$? If $x \in \mathcal{A}$, then clearly yes. Since $x \in \mathcal{A}\|$ implies that $\psi_{\mathcal{A}}(x) \neq o$ (because $o \prec \rho$), and since

$$\psi_{\mathcal{A}}(x) = o \iff \{\eta \in \ell : \tau_{-\eta}(x) \in \mathcal{A}\} = \emptyset,$$

we see that $x \in \mathcal{A}\| \Longrightarrow \{\eta \in \ell : \tau_{-\eta}(x) \in \mathcal{A}\} \neq \emptyset$. Thus, if $x \in \mathcal{A}\| \setminus \mathcal{A}$, then can it happen that $\rho \not\preceq \eta$ for all members of the nonempty set $\{\eta \in \ell : \tau_{-\eta}(x) \in \mathcal{A}\}$? The answer here is yes. Indeed, there are important examples in section 4 of the abstraction we are considering for which no two distinct elements of ℓ are $\preceq$-comparable. In this case, it therefore follows that $\rho \preceq \eta$ for an $\eta \in \ell$ if and only if $\eta = \rho$, i.e., if and only if $x \in \mathcal{A}$. Let us here repeat Lemma 16 adjusted for the case at hand.

Lemma 18 *We have the following for all $a, b \in \mathcal{L}$.*

1. $[a, b]\| \subset [a, e] \cap [o, b]\|$ *(equivalently, $\sigma_{ab} \preceq_p \varepsilon_a \wedge \widetilde{\delta}_b$).*
2. *If $[o, b]$ is a kernel set, then $[a, b]$ is a kernel set and $\sigma_{ab} = \varepsilon_a \wedge \widetilde{\delta}_b$.*

We will presently see, in Example 1 (sect. 4.1), a morphologically important complete lattice for which $[o, b]$ is not generally a kernel set but $\sigma_{ab} = \varepsilon_a \wedge \widetilde{\delta}_b$ holds for all a and b (cf. Thm. 8). We will see an equally important lattice in Example 2 (sect. 4.2), for which $[o, b]$ is a kernel set for all b; and, finally, in Example 3 (sect. 4.3), we will detail a case of Lemma 16, in which every b belongs to $[O, \mathcal{K}_1]$.

4 Examples

4.1 Example 1: $\mathcal{L} = \mathrm{ERV}(\Re^n)$

Let $\mathrm{ERV}(\Re^n)$ denote the set of extended real valued (ERV) functions defined on the n-fold cartesian product of the real continuum $\Re$, where n is a positive integer.

Definition 38 *If $f, g \in \mathrm{ERV}(\Re^n)$, then define*

$$f \preceq g \iff f(x) \leq g(x) \text{ for all } x \in \Re^n.$$

Proposition 33 $(\mathrm{ERV}(\Re^n), \preceq)$ *is a complete poset.*

Proof It is clear that $(\mathrm{ERV}(\Re^n), \preceq)$ is a poset. If $\{f_\alpha\}$ is any subset of $\mathrm{ERV}(\Re^n)$, then the supremum $\bigvee_\alpha f_\alpha$ and infimum $\bigwedge_\alpha f_\alpha$ of $\{f_\alpha\}$ are given respectively by the pointwise supremum and infimum; i.e., for each $x \in \Re^n$ we have

$$\left(\bigvee_\alpha f_\alpha\right)(x) = \sup_\alpha f_\alpha(x) \quad \text{and} \quad \left(\bigwedge_\alpha f_\alpha\right)(x) = \inf_\alpha f_\alpha(x).$$

This is clear because $\sup_\alpha f_\alpha(x)$ $(x \in \Re^n)$ is an ERV function on $\Re^n$ that is plainly the least upper bound of $\{f_\alpha\}$; and likewise for $\inf_\alpha f_\alpha(x)$.

Corollary 20 *The lattice operations in* $(\mathrm{ERV}(\Re^n), \preceq)$ *are the pointwise supremum and infimum; that is, if $f, g \in \mathrm{ERV}(\Re^n)$ and $x \in \Re^n$, then $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$.*

Definition 39 *For each $x \in \Re^n$ and $t \in \Re$, define $\eta_{x,t}, \omega_{x,t} \in \mathrm{ERV}(\Re^n)$ by*

$$\eta_{x,t}(y) = \begin{cases} -\infty & \text{if } y \neq x \\ t & \text{if } y = x \end{cases} \quad \text{and} \quad \omega_{x,t}(y) = \begin{cases} \infty & \text{if } y \neq x \\ t & \text{if } y = x \end{cases}.$$

Denote the sets $\{\eta_{x,t} : x \in \Re^n, t \in \Re\}$ and $\{\omega_{x,t} : x \in \Re^n, t \in \Re\}$ by $\mathcal{X}$ and $\mathcal{N}$, respectively.

Proposition 34 *$\mathcal{X}$ $(\mathcal{N})$ is a sup-generating (inf-generating) subset of* $\mathrm{ERV}(\Re^n)$.

Proof If $f \in \mathrm{ERV}(\Re^n)$, then it is clear that $\sup\{\eta \in \mathcal{X} : \eta \preceq f\} \preceq f$. For each $x \in \Re^n$ such that $f(x)$ is not infinite, moreover, we have $\eta_{x,f(x)} \in \mathcal{X}$, $\eta_{x,f(x)} \preceq f$, and $\eta_{x,f(x)}(x) = f(x)$. If $f(x) = \infty$, let $\{t_i\}$ be an increasing sequence of reals tending to ∞, and note that $\{\eta_{x,t_i}\} \subset \mathcal{X}$. Clearly, then, $f = \sup\{\eta \in \mathcal{X} : \eta \preceq f\}$. The rest is similar.

Remark 25 *$\eta_{x,t}$ is $\preceq$-comparable with $\eta_{y,r} \iff x = y$.*

Definition 40 *For each $x \in \Re^n$, $t \in \Re$, and $f \in \mathrm{ERV}(\Re^n)$, let $f + (x,t) \in \mathrm{ERV}(\Re^n)$ be defined for all $y \in \Re^n$ by $(f + (x,t))(y) = f(y - x) + t$.*

Definition 41 *Let $\mathbf{R}_{n+1}$ denote the commutative group $(\Re^{n+1}, +)$, where $+$ denotes vector addition in $\Re^{n+1}$, and denote the elements of $\mathbf{R}_{n+1}$ by (x,t) where $x \in \Re^n$ and $t \in \Re$. For each $(x,t) \in \mathbf{R}_{n+1}$ and $f \in \mathrm{ERV}(\Re^n)$, let $s_{x,t} : f \longmapsto f + (x,t)$ and define the mapping $s : \mathbf{R}_{n+1} \times \mathrm{ERV}(\Re^n) \longrightarrow \mathrm{ERV}(\Re^n)$ by $s((x,t), f) = s_{x,t}(f) = f + (x,t)$.*

Proposition 35 *$(\mathbf{R}_{n+1}, s)$ is an effective group action on $\mathrm{ERV}(\Re^n)$ and $s_{x,t}$ is a lattice automorphism of $\mathrm{ERV}(\Re^n)$ for each $(x,t) \in \mathbf{R}_{n+1}$. Moreover, $(\mathbf{R}_{n+1}, s)$ is $\mathcal{X}$-admissable.*

Proof $(\mathbf{R}_{n+1}, s)$ is a group action on $\mathrm{ERV}(\Re^n)$ because $(x,t) \longmapsto s_{x,t}$ has the homomorphism property $(x,t) + (\xi,\tau) \longmapsto s_{(x,t)+(\xi,\tau)} = s_{\xi,\tau} \circ s_{x,t}$, as is easily verified. The action is effective because distinct (x,t) give rise to distinct $s_{x,t}$, i.e., the homomorphism is bijective and therefore an isomorphism. If $(x,t) \in \mathbf{R}_{n+1}$ and $f, g \in \mathrm{ERV}(\Re^n)$, then one easily verifies that

$$s_{x,t}(f \wedge g) = s_{x,t}(f) \wedge s_{x,t}(g) \quad \text{and} \quad s_{x,t}(f \vee g) = s_{x,t}(f) \vee s_{x,t}(g),$$

i.e., each $s_{x,t}$ is a lattice automorphism of $\mathrm{ERV}(\Re^n)$. For the $\mathcal{X}$-admissability of the action, let $\eta_{\xi,\tau} \in \mathcal{X}$ and note that $s_{x,t}(\eta_{\xi,\tau}) = \eta_{\xi,\tau} + (x,t) = \eta_{\xi+x,\tau+t} \in \mathcal{X}$; moreover, given $\eta_{x,t}, \eta_{y,r} \in \mathcal{X}$ one readily sees that $s_{y-x,r-t}(\eta_{x,t}) = \eta_{y,r}$. This completes the proof.

Remark 26 *A mapping $\psi : \mathrm{ERV}(\Re^n) \longrightarrow \mathrm{ERV}(\Re^n)$ is $(\mathbf{R}_{n+1}, s)$-invariant (or more briefly $\mathbf{R}_{n+1}$-invariant) if and only if $\psi(f + (x,t)) = \psi(f) + (x,t)\ \forall\ f \in \mathrm{ERV}(\Re^n)$ and $(x,t) \in \mathbf{R}_{n+1}$.*

Definition 42 *To define the kernels of $\mathbf{R}_{n+1}$-invariant operators on $\mathrm{ERV}(\Re^n)$, we choose $\eta_{\mathbf{0},0}$ as the reference element of $\mathcal{X}$.*

Accordingly, if ψ is an $\mathbf{R}_{n+1}$-invariant operator on $\mathrm{ERV}(\Re^n)$, then $\mathcal{V}(\psi)$ is given by

$$\mathcal{V}(\psi) = \{f \in \mathrm{ERV}(\Re^n) : \eta_{\mathbf{0},0} \preceq \psi(f)\}.$$

Since $-\infty = \eta_{\mathbf{0},0}(x) \preceq \psi(f)(x)$ for all $x \neq \mathbf{0}$, it follows that

$$\mathcal{V}(\psi) = \{f \in \mathrm{ERV}(\Re^n) : 0 \leq \psi(f)(\mathbf{0})\}.$$

Definition 43 *A subset $\mathcal{V}$ of $\mathrm{ERV}(\Re^n)$ will be called a* **k-set** *if*

$$f \in \mathcal{V} \iff f + (\mathbf{0}, t) \in \mathcal{V} \text{ for all } t > 0.$$

Proposition 36 *If ψ is an $\mathbf{R}_{n+1}$-invariant operator on $\mathrm{ERV}(\Re^n)$, then $\mathcal{V}(\psi)$ is a k-set.*

Proof If $f \in \mathcal{V}(\psi)$, then $0 \leq \psi(f)(\mathbf{0})$. If $t > 0$, then $\psi(f + (\mathbf{0},t))(\mathbf{0}) = \psi(f)(\mathbf{0}) + t > 0$, by the $\mathbf{R}_{n+1}$-invariance of ψ. Hence we have shown that

$$f \in \mathcal{V}(\psi) \Longrightarrow f + (\mathbf{0},t) \in \mathcal{V}(\psi) \text{ for all } t > 0.$$

If, on the other hand, $f + (\mathbf{0},t) \in \mathcal{V}(\psi)$ for all $t > 0$, then

$$0 < \psi(f + (\mathbf{0},t))(\mathbf{0}) = \psi(f)(\mathbf{0}) + t \text{ for all } t > 0.$$

Clearly, then, $\psi(f)(\mathbf{0}) \geq 0$, and it follows that $f \in \mathcal{V}(\psi)$.

Proposition 37 *If ψ is an $\mathbf{R}_{n+1}$-invariant operator on* $\mathrm{ERV}(\Re^n)$ *with kernel $\mathcal{V}(\psi)$, then for all $f \in \mathrm{ERV}(\Re^n)$ and $x \in \Re^n$ it follows that*

$$\psi(f)(x) = \sup\{t : f - (x,t) \in \mathcal{V}(\psi)\}.$$

Proof For all $f \in \mathrm{ERV}(\Re^n)$, it follows from general considerations that

$$\psi(f) = \bigvee\{\eta_{y,t} : (y,t) \in \mathbf{R}_{n+1}, f - (y,t) \in \mathcal{V}(\psi)\}.$$

For each $x \in \Re^n$ we therefore get the desired result

$$\psi(f)(x) = \sup\{t : f - (x,t) \in \mathcal{V}(\psi)\}.$$

Proposition 38 *If $\mathcal{V} \subset \mathrm{ERV}(\Re^n)$, then the operator ψ defined for all $f \in \mathrm{ERV}(\Re^n)$ by*

$$\psi(f)(x) = \sup\{t : f - (x,t) \in \mathcal{V}\} \quad (x \in \Re^n)$$

is $\mathbf{R}_{n+1}$-invariant; moreover, if $\mathcal{V}$ is a k-set, then $\mathcal{V}(\psi) = \mathcal{V}$.

Proof To verify $\mathbf{R}_{n+1}$-invariance we evaluate

$$\psi(f + (y,r))(x) = \sup\{t : f + (y - x, r - t) \in \mathcal{V}\}$$

and compare the result with

$$(\psi(f) + (y,r))(x) = \psi(f)(x - y) + r = \sup\{\tau + r : f + (y - x, -\tau) \in \mathcal{V}\}.$$

Setting $\tau + r = t$ this becomes

$$(\psi(f) + (y,r))(x) = \psi(f)(x - y) + r = \sup\{t : f + (y - x, r - t) \in \mathcal{V}\}.$$

Thus ψ is $\mathbf{R}_{n+1}$-invariant independently of the k-set character of $\mathcal{V}$.

For the second part, we begin with

$$\mathcal{V}(\psi) = \{g \in \mathrm{ERV}(\Re^n) : 0 \leq \sup\{r : g - (\mathbf{0},r) \in \mathcal{V}\}\}.$$

Since it is clear that $g \in \mathcal{V} \Longrightarrow \sup\{r : g - (\mathbf{0},r) \in \mathcal{V}\} \geq 0$, we see that $\mathcal{V} \subset \mathcal{V}(\psi)$, again independently of the k-set character of $\mathcal{V}$. Suppose finally that $h \in \mathcal{V}(\psi)$, so that $0 \leq \sup\{r : h - (\mathbf{0},r) \in \mathcal{V}\}$, i.e., $h - (\mathbf{0},r_i) \in \mathcal{V}$ for all r_i of a real sequence $\{r_i\}$ such that $r_i \uparrow 0$. If $h \notin \mathcal{V}$, then there is a $t > 0$ such that $h - (\mathbf{0},-t) \notin \mathcal{V}$, because $\mathcal{V}$ is a k-set. But for i sufficiently large it follows that $h - (\mathbf{0},r_i) \in \mathcal{V}$ and $-t < r_i \leq 0$. Thus there is an i such that $r_i + t > 0$ and therefore, again by the k-set character of $\mathcal{V}$, $h + (\mathbf{0},t) \in \mathcal{V}$. This contradiction completes the proof.

Corollary 21 *A subset $\mathcal{V}$ of $\mathrm{ERV}(\Re^n)$ is a kernel set if and only if $\mathcal{V}$ is a k-set.*

Definition 44 *If $f \in \mathrm{ERV}(\Re^n)$ and $\mathcal{V} \subset \mathrm{ERV}(\Re^n)$, then we define the following.*

1. $S(f) = \{x \in \Re^n : f(x) > -\infty\}$.
2. $f \uparrow = \{f + (\mathbf{0}, t) : t \geq 0\}$.
3. $f \downarrow = \{f + (\mathbf{0}, t) : t < 0\}$.
4. $f \updownarrow = \{f + (\mathbf{0}, t) : t \in \Re\}$.
5. $\mathbf{B}(\mathcal{V}) = \{\inf(f \updownarrow \cap \mathcal{V}) : f \in \mathcal{V}\}$.

Remark 27 $\{f \updownarrow : f \in \mathrm{ERV}(\Re^n)\}$ *is a partition of* $\mathrm{ERV}(\Re^n)$.

Lemma 19 *If $h \in \mathrm{ERV}(\Re^n)$ and $\mathcal{A}$ is a nonempty subset of $h \updownarrow$, then the following hold.*

1. *There is a unique nonempty subset A of $\Re$ such that $\mathcal{A} = \{h + (\mathbf{0}, r) : r \in A\}$.*
2. *If A is bounded from below and $g = \inf \mathcal{A}$, then there is a sequence $\{t_i\}$ of positive reals such that $t_i \downarrow 0$ and $g + (\mathbf{0}, t_i) \in \mathcal{A}$ for all i.*

Proof (1) Each element of $\mathcal{A}$ is of the form $h + (\mathbf{0}, r)$ for a unique real r, i.e., there is a unique nonempty subset A of $\Re$ such that $\mathcal{A} = \{h + (\mathbf{0}, r) : r \in A\}$.

(2) For all $x \in \Re^n$ we have that $g(x) = (\inf \mathcal{A})(x) = \inf\{h(x) + r : r \in A\}$. Since A is bounded from below, it follows that $g(x) = h(x) + \inf A$ for all $x \in \Re^n$; hence, $g = h + (\mathbf{0}, \inf A)$. There is a sequence $\{r_i\}$ in A such that $r_i \downarrow \inf A$. Thus if we let $t_i = r_i - \inf A$, then $\{t_i\}$ is a sequence of positive reals such that $t_i \downarrow 0$, and for all i

$$g + (\mathbf{0}, t_i) = h + (\mathbf{0}, t_i + \inf A) = h + (\mathbf{0}, r_i) \in \mathcal{A}.$$

Definition 45 *If $A \subset \Re^n$, then define $\aleph_A \in \mathrm{ERV}(\Re^n)$ by*

$$\aleph_A(y) = \begin{cases} \infty & \text{if } y \in A \\ -\infty & \text{if } y \notin A \end{cases}.$$

If $\aleph \in \mathrm{ERV}(\Re^n)$ satisfies $\aleph = \aleph_A$ for some $A \subset \Re^n$, then $\aleph$ will be called an **aleph function**.

Lemma 20 *If $\aleph$ is an aleph function, then we have the following.*

1. $\aleph \updownarrow = \{\aleph\}$.
2. $\aleph \in \mathcal{V} \Longrightarrow \aleph \in \mathbf{B}(\mathcal{V})$.
3. $\aleph \in \mathbf{B}(\mathcal{V}) \Longrightarrow \exists\, h \in \mathcal{V}$ *such that* $\{x : h(x) = \infty\} = \{x : \aleph(x) = \infty\}$.

Proof (1) and (2) are trivial. (3) If $\aleph \in \mathbf{B}(\mathcal{V})$, then, by definition, there is an $h \in \mathcal{V}$ such that $\inf(h \updownarrow \cap \mathcal{V}) = \aleph$. Since $h \updownarrow \cap \mathcal{V} \neq \emptyset$, there is a unique nonempty subset A of $\Re$ such that $h \updownarrow \cap \mathcal{V} = \{h+(\mathbf{0}, r) : r \in A\}$. If A is not bounded from below, then there is a sequence $\{r_i\}$ in A such that $r_i \downarrow -\infty$, and this implies that $\{x : h(x) = \infty\} = \{x : \aleph(x) = \infty\}$. Hence in this case, h fulfills the conditions in (3). If A is bounded from below, then by Lemma 19 there is a positive sequence $t_i \downarrow 0$ such that $\aleph + (\mathbf{0}, t_i) \in h \updownarrow \cap \mathcal{V}$ for all i; but this implies that $\aleph \in \mathcal{V}$. Hence in this case, $\aleph$ itself fulfills the conditions in (3).

Lemma 21 *If $\mathcal{V} \subset \mathrm{ERV}(\Re^n)$, $g \in \mathbf{B}(\mathcal{V})$, and g is not an aleph function, then $g \in \mathcal{V}\|$.*

Proof Let $g \in \mathbf{B}(\mathcal{V})$. Then there is an $h \in \mathcal{V}$ such that $g = \inf(h \updownarrow \cap \mathcal{V})$. It is clear that $h \updownarrow \cap \mathcal{V} \neq \emptyset$. Moreover, if the nonempty set $A \subset \Re$ such that

$$h \updownarrow \cap \mathcal{V} = \{h + (\mathbf{0}, r) : r \in A\}$$

were not bounded from below, then we would have $g(x) = -\infty$ when $h(x) \neq \infty$ and $g(x) = h(x)$ when $h(x) = \infty$, i.e., g would be an aleph function. We may therefore assume that A is bounded from below. There is thus a sequence $\{t_i\}$ of positive reals such that $t_i \downarrow 0$ and $g + (\mathbf{0}, t_i) \in \mathcal{V}$ for all i. Therefore, $g + (\mathbf{0}, t) \in \mathcal{V}\|$ for all $t > 0$, and this implies the desired conclusion that $g \in \mathcal{V}\|$.

Definition 46 *For each $\mathcal{V} \subset \mathrm{ERV}(\Re^n)$, let $\mathbf{B}_{-\aleph}(\mathcal{V})$ denote $\mathbf{B}(\mathcal{V})$ with its aleph functions removed; also, let $\mathcal{V}_\aleph$ denote the set of aleph functions in $\mathcal{V}$. Then we define $\widetilde{\mathbf{B}}(\mathcal{V})$ by*

$$\widetilde{\mathbf{B}}(\mathcal{V}) = \mathbf{B}_{-\aleph}(\mathcal{V}) \cup \mathcal{V}_\aleph \subset \mathbf{B}(\mathcal{V}).$$

Proposition 39 *If $\mathcal{V} \subset \mathrm{ERV}(\Re^n)$, then the following hold.*

1. $\mathcal{V} = \bigcup_{f \in \mathbf{B}(\mathcal{V})} f \updownarrow \cap \mathcal{V}$.
2. $\mathcal{V}\| = \bigcup_{f \in \widetilde{\mathbf{B}}(\mathcal{V})} \{f + (\mathbf{0}, t) : t \geq 0\}$.

Proof (1) It is clear that $\mathcal{V} = \bigcup\{f \updownarrow \cap \mathcal{V} : f \in \mathrm{ERV}(\Re^n)\}$. Moreover, if $f \updownarrow \cap \mathcal{V} \neq \emptyset$, then there is a $g \in \mathcal{V}$ such that $g \updownarrow = f \updownarrow$. Therefore, $h = \inf(f \updownarrow \cap \mathcal{V}) \in \mathbf{B}(\mathcal{V})$ and $h \updownarrow = f \updownarrow$.

(2) Put $\mathcal{V}_\uparrow = \bigcup_{f \in \widetilde{\mathbf{B}}(\mathcal{V})} \{f + (\mathbf{0}, t) : t \geq 0\}$. Since $f \updownarrow \cap \mathcal{V} \subset \{f + (\mathbf{0}, t) : t \geq 0\}$ for all $f \in \widetilde{\mathbf{B}}(\mathcal{V})$, it follows that $\mathcal{V} \subset \mathcal{V}_\uparrow$. If $g \in \mathcal{V}_\uparrow$, then there is an $f \in \widetilde{\mathbf{B}}(\mathcal{V})$ and a $\tau \geq 0$ such that $g = f + (\mathbf{0}, \tau)$; hence $g + (\mathbf{0}, t) = f + (\mathbf{0}, \tau + t) \in \mathcal{V}_\uparrow$ for all $t > 0$. On the other hand, if $g + (\mathbf{0}, t) \in \mathcal{V}_\uparrow$ for all $t > 0$, then there is an $f \in \widetilde{\mathbf{B}}(\mathcal{V})$ such that $\{g + (\mathbf{0}, t) : t > 0\} \subset \{f + (\mathbf{0}, r) : r \geq 0\}$. Thus it follows that $g = f + (\mathbf{0}, r')$ for some $r' \geq 0$; hence $g \in \mathcal{V}_\uparrow$ and we see that $\mathcal{V}_\uparrow$ is a kernel set that contains $\mathcal{V}$. Thus it remains to prove that $\mathcal{V}'$ is a kernel set and contains $\mathcal{V} \Longrightarrow \mathcal{V}_\uparrow \subset \mathcal{V}'$.

If $g \in \mathcal{V}_\uparrow$, then there is an $f \in \widetilde{\mathbf{B}}(\mathcal{V})$ such that $g = f + (\mathbf{0}, t)$ for some $t \geq 0$. We consider the two cases (a) $t > 0$ and (b) $t = 0$ separately. (a) Since $f \in \widetilde{\mathbf{B}}(\mathcal{V})$, there is an $h \in \mathcal{V}$ such that $f = \inf(h \updownarrow \cap \mathcal{V})$. Thus $h = f + (\mathbf{0}, r')$ for some $r' \geq 0$. In fact we can choose h so that $0 < r' < t$, i.e., so that $h \prec g$. Since $\mathcal{V}'$ is a kernel set containing $\mathcal{V}$, it follows that $h+(\mathbf{0}, r) = f+(\mathbf{0}, r'+r) \in \mathcal{V}'$ for all $r > 0$. Consequently, $h + (\mathbf{0}, r) = g + (\mathbf{0}, (r' - t) + r) \in \mathcal{V}'$ for all $r > 0$. Since $r' - t < 0$, it is clear in this case that $g \in \mathcal{V}'$ (choose $r = t - r'$). (b) If $t = 0$, then $f = g$ and we see that $g \in \widetilde{\mathbf{B}}(\mathcal{V})$. If g is not an aleph function, then $g \in \mathcal{V}\|$, by Lemma 21. Since $\mathcal{V}' \supset \mathcal{V}\|$, it therefore follows that $g \in \mathcal{V}'$. On the other hand. if g is an aleph function, then we must have that $g \in \mathcal{V} \subset \mathcal{V}\| \subset \mathcal{V}'$. This completes the proof.

Proposition 40 *If $[g, h] \subset \mathrm{ERV}(\Re^n)$, then the following hold.*

1. *If $[g, h] \neq \emptyset$, then $[g, h]$ is a kernel set if and only if $h = \aleph_A$ for some $A \subset \Re^n$.*

2. *$[g, h]\| = \{f \uparrow : f \in [g, h]\} \subset [g, \aleph_{S(h)}]$.*

3. *If h is not an aleph function, then the containment $[g, h]\| \subset [g, \aleph_{S(h)}]$ is proper.*

Proof (1) If $[g, h]$ is a kernel set, then it follows that $h(x) + t \leq h(x)$ for all $x \in \Re^n$ and all $t > 0$; hence it is clear that h is an aleph function. On the other hand, if $[g, h] \neq \emptyset$, $h = \aleph_A$ for some $A \subset \Re^n$, and $f \in [g, h]$, then it is clear that $f + (\mathbf{0}, t) \in [g, h]$ for all $t > 0$. To see that $f + (\mathbf{0}, t) \in [g, h]$ for all $t > 0 \Longrightarrow f \in [g, h]$, first note that $f \in \mathrm{ERV}(\Re^n)$ and $f + (\mathbf{0}, t) \in [g, h]$ for all $t > 0$ clearly imply that $f \preceq h$ and $f = -\infty$ on A^c. Moreover, if there were an $x \in A$ such that $f(x) < g(x)$, then there would be a $t > 0$ such that $f(x) + t < g(x)$, i.e., a $t > 0$ such that $g \not\preceq f + (\mathbf{0}, t)$.

(2) If h is an aleph function, then $[g, h]\| = [g, \aleph_A]$ for some $A \subset \Re^n$ and it is clear that $[g, \aleph_A] = \{f \uparrow : f \in [g, \aleph_A]$. Assume, then, that h is not an aleph function. First we determine $\mathbf{B}([g, h])$. For each $f \in [g, h]$ we define

$$\tau_f = \sup\{\tau : g \preceq f - (\mathbf{0}, \tau)\}$$

and note that $(\inf(f \updownarrow \cap [g, h]))(x) = \inf\{f(x) + t : f + (\mathbf{0}, t) \in [g, h]\}$ for each $x \in \Re^n$. Thus if $\tau_f < \infty$, then $\inf(f \updownarrow \cap [g, h]) = f - (\mathbf{0}, \tau_f) \in [g, h]$, and it follows that $\{f \uparrow : f \in [g, h] \text{ and } \tau_f < \infty\} \subset [g, h]\|$. If $\tau_f = \infty$, then $g \preceq f - (\mathbf{0}, \tau)$ for all positive τ and for each each $x \in \Re^n$ we have

$$\inf(f \updownarrow \cap [g, h])(x) = \inf\{f(x) - \tau : g \preceq f - (\mathbf{0}, \tau)\}.$$

If $f(x) \notin \Re$, then $\inf(f \updownarrow \cap [g, h])(x) = f(x)$; otherwise, $\inf(f \updownarrow \cap [g, h])(x) = -\infty$. Thus $\inf(f \updownarrow \cap [g, h]) = \aleph_{\{x : f(x) = \infty\}}$. Define $A, B, C \subset \Re^n$ by $A = \{x : h(x) = -\infty\}$, $B = \{x : h(x) \in \Re\}$, and $C = \{x : h(x) = \infty\}$. Since $f \preceq h$, it follows that $\{x : f(x) = \infty\} \subset C$ and hence that $\aleph_{\{x : f(x) = \infty\}} \preceq h$. Furthermore, it is clear that $g(x) = \aleph_{\{x : f(x) = \infty\}}$ for all $x \in A$. Since h is not an aleph function (i.e., $B \neq \emptyset$), it follows that $g \preceq \aleph_{\{x : f(x) = \infty\}} \iff g(x) = -\infty$ for all $x \in B$. Thus if $g(x) > -\infty$ for some $x \in B$, then it follows by Proposition 39 that

$$[g, h]\| = \{f \uparrow : f \in [g, h], \tau_f < \infty\} \cup \{f \updownarrow : f \in [g, h], \tau_f = \infty\}.$$

This conclusion is not changed if $g = -\infty$ on B; indeed,

$$\{\aleph_{\{x:f(x)=\infty\}} : f \in [g,h], \tau_f = \infty\} \subset [g,h]$$

in this case, and so $\{\aleph_{\{x:f(x)=\infty\}} : f \in [g,h], \tau_f = \infty\} \subset \{f \updownarrow: f \in [g,h], \tau_f = \infty\}$ (because $\aleph_{\{x:f(x)=\infty\}} \updownarrow = \aleph_{\{x:f(x)=\infty\}}$). For the rest, let us note the following:

(a) $f \in [g,h]$ and $\tau_f < \infty \Longrightarrow f \uparrow \in [g,h]\|$.

(b) $f \in [g,h]$ and $\tau_f = \infty \Longrightarrow f \uparrow \in [g,h]\|$.

(c) $\tilde{f} \in [g,h]\| \Longrightarrow \tilde{f} \in \{f \uparrow: f \in [g,h], \tau_f < \infty\} \cup \{f \updownarrow: f \in [g,h], \tau_f = \infty\}$.

(d) $\tilde{f} \in \{f \uparrow: f \in [g,h], \tau_f < \infty\} \Longrightarrow \tilde{f} \in \{f \uparrow: f \in [g,h]\}$.

To prove that $[g,h]\| = \{f \uparrow: f \in [g,h]\}$ it therefore remains to show that

$$\tilde{f} \in \{f \updownarrow: f \in [g,h], \tau_f = \infty\} \Longrightarrow \tilde{f} \in \{f \uparrow: f \in [g,h]\}.$$

But this is clear because every function in $\{f \downarrow: f \in [g,h], \tau_f = \infty\}$ lies in $[g,h]$. Since $[g,h]\| \subset [g, \aleph_{S(h)}]$ is obvious, this completes the proof of (2).

(3) If h is not an aleph function, then $\aleph_{S(h)} \notin [g,h]$.

Corollary 22 *Let $f, g, h \in \mathrm{ERV}(\Re^n)$, where $g \preceq h$, and assume that*

1. *h is not an aleph function.*
2. *$f \in [g, \aleph_{S(h)}]$.*
3. *For each $t > 0$, there is an $x \in S(h) \setminus \{x : h(x) = \infty\}$ such that $h(x) + t < f(x)$.*

Then $f \in [g, \aleph_{S(h)}] \setminus [g,h]\|$.

Proof Assume on the contrary that $f \in [g,h]\|$. Then there is a $\varphi \in [g,h]$ and a $t > 0$ such that $f = \varphi + (\mathbf{0}, t)$. Let $x \in S(h) \setminus \{x : h(x) = \infty\}$ be such that $h(x) + t < f(x)$. Since $\varphi(x) \leq h(x)$, it follows from $f = \varphi + (\mathbf{0}, t)$ that $f(x) \leq h(x) + t$, which contradicts $h(x) + t < f(x)$. Hence $f \notin [g,h]\|$ and this completes the proof.

Remark 28 *If $g, h \in \mathrm{ERV}(\Re^n)$, then the operators ε_g (erosion by g), $\tilde{\delta}_h$, and σ_{gh} are given for all $f \in \mathrm{ERV}(\Re^n)$ and $x \in \Re^n$ by the following.*

1. $\varepsilon_g(f)(x) = \sup\{t : g \preceq f - (x,t)\}$.
2. $\tilde{\delta}_h(f)(x) = \sup\{t : f - (x,t) \preceq h\}$.
3. $\sigma_{gh}(f)(x) = \sup\{t : g \preceq f - (x,t) \preceq h\}$.

Remark 29 *If $g, h \in \mathrm{ERV}(\Re^n)$ and $g \preceq h$, then we have the following.*

1. $\{f \in [g,h] : \tau_f < \infty\} = \{f \in [g,h] : \varepsilon_g(f)(\mathbf{0}) < \infty\}$.

2. $\{f \in [g,h] : \tau_f = \infty\} = \{f \in [g,h] : \varepsilon_g(f)(\mathbf{0}) = \infty\}$.

Remark 30 *If $g \in \mathrm{ERV}(\Re^n)$, then $\mathcal{V}(\varepsilon_g) = [g, \aleph_{\Re^n}]$, there is a unique operator ς_g (dilation by g) on $\mathrm{ERV}(\Re^n)$ whose upper adjoint is ε_g (i.e., $(\varepsilon_g, \varsigma_g)$ is an adjunction), and $\forall\, f \in \mathrm{ERV}(\Re^n)$*

$$\varsigma_g(f) = \inf\{u \in \mathrm{ERV}(\Re^n) : f \preceq \varepsilon_g(u)\}.$$

Proposition 41 *If $g, h \in \mathrm{ERV}(\Re^n)$, then $\mathcal{V}(\sigma_{gh}) = \mathcal{V}(\varepsilon_g) \cap \mathcal{V}(\widetilde{\delta}_h)$.*

Proof By Lemma 18, $\mathcal{V}(\sigma_{gh}) \subset \mathcal{V}(\varepsilon_g) \cap \mathcal{V}(\widetilde{\delta}_h) = [g, \aleph_{\Re^n}] \cap [\aleph_\emptyset, h]\|$. If $\varphi \in \mathcal{V}(\sigma_{gh}) = [g,h]\|$, then according to Proposition 40 there is a $\phi \in [g,h]$ and a $t \geq 0$ such that $\varphi = \phi \dot{+} (\mathbf{0}, t)$. Thus it is clear that $\varphi \in [g, \aleph_{\Re^n}]$. Since $[\aleph_\emptyset, h]\| = \{f \uparrow : f \in [\aleph_\emptyset, h]\}$ (again by Proposition 40) and $\phi \in [\aleph_\emptyset, h]$, it is also clear that $\varphi \in [\aleph_\emptyset, h]\|$.

Corollary 23 *If ψ is an $\mathbf{R}_{n+1}$-invariant operator on $\mathrm{ERV}(\Re^n)$, then the Ronse decomposition $\psi = \sup\{\sigma_{gh} : [g,h] \subset \mathcal{V}(\psi)\}$ (where the supremum is relative to $\mathcal{O}_{\mathbf{R}_{n+1}}$) can be written $\psi = \sup\{\varepsilon_g \wedge \widetilde{\delta}_h : [g,h] \subset \mathcal{V}(\psi)\}$.*

Theorem 10 *We summarize several previous results as follows.*

1. $\mathbf{\Lambda}_{\mathbf{R}_{n+1}} = \{\sigma_{gh} : h = \aleph_A, A \in \mathcal{P}(\Re^n)\}$.

2. $\mathcal{E}_{\mathbf{R}_{n+1}} = \{\varepsilon_g : g \in \mathrm{ERV}(\Re^n)\}$.

3. $\widetilde{\mathcal{D}}_{\mathbf{R}_{n+1}} = \{\widetilde{\delta}_h : h = \aleph_A, A \in \mathcal{P}(\Re^n)\}$.

Thus if ψ is an $\mathbf{R}_{n+1}$-invariant operator on $\mathrm{ERV}(\Re^n)$, then the Ronse decomposition

$$\psi = \sup\{\varepsilon_g \wedge \widetilde{\delta}_h : [g,h] \subset \mathcal{V}(\psi)\}$$

includes terms $\varepsilon_g \wedge \widetilde{\delta}_h \notin \mathbf{\Lambda}_{\mathbf{R}_{n+1}}$, i.e., such that $\widetilde{\delta}_h$ is not an anti-dilation (cf. Thm. 8).

4.2 Example 2: $\mathcal{L} = \mathcal{P}(\Re^m)$

Next we consider the powerset $\mathcal{P}(\Re^m)$ (m a positive integer) with the ordering relation $\subset$ and the associated lattice operations $\cap$ and $\cup$. It is well known that $(\mathcal{P}(\Re^m), \subset)$ is complete.

Remark 31 *If $\mathcal{A} \subset \mathcal{P}(\Re^m)$, then $\sup \mathcal{A} = \bigcup\{A : A \in \mathcal{A}\}$ and $\inf \mathcal{A} = \bigcap\{A : A \in \mathcal{A}\}$.*

It is also well known that the set ℓ of singleton subsets of $\Re^m$ is a sup-generating subset of $\mathcal{P}(\Re^m)$; indeed, every subset of $\Re^m$ is the union of its points.

Definition 47 *Let $\mathbf{R}_m$ denote the commutative group $(\Re^m, +)$ where $+$ denotes vector addition in $\Re^m$. For each $x \in \mathbf{R}_m$ (i.e., for each $x \in \Re^m$) and $A \in \mathcal{P}(\Re^m)$, let*

$$\sigma_x(A) = A + x \equiv \{a + x : a \in A\}$$

and define the mapping $\sigma : \mathbf{R}_m \times \mathcal{P}(\Re^m) \longrightarrow \mathcal{P}(\Re^m)$ by $\sigma(x, A) = A + x$.

Proposition 42 *$(\mathbf{R}_m, \sigma)$ is an effective group action on $\mathcal{P}(\Re^m)$, σ_x is an automorphism of $\mathcal{P}(\Re^m)$ for all $x \in \mathbf{R}_m$, and $(\mathbf{R}_m, \sigma)$ acts ℓ-admissably on $\mathcal{P}(\Re^m)$.*

Remark 32 *A mapping $\psi : \mathcal{P}(\Re^m) \longrightarrow \mathcal{P}(\Re^m)$ is $(\mathbf{R}_m, \sigma)$-invariant (or more briefly $\mathbf{R}_m$-invariant) if and only if $\psi(A + x) = \psi(A) + x$ for all $A \in \mathcal{P}(\Re^m)$ and $x \in \mathbf{R}_m$.*

Definition 48 *To define the kernels of $\mathbf{R}_m$-invariant operators on $\mathcal{P}(\Re^m)$, let the origin $\mathbf{0}$ of $\Re^m$ be the reference element of ℓ.*

Accordingly, if ψ is an $\mathbf{R}_m$-invariant operator on $\mathcal{P}(\Re^m)$, then the kernel of ψ is given by

$$\mathcal{V}(\psi) = \{A \in \mathcal{P}(\Re^m) : \mathbf{0} \in \psi(A)\}.$$

Proposition 43 *If ψ is an $\mathbf{R}_m$-invariant operator on $\mathcal{P}(\Re^m)$ with kernel $\mathcal{V}(\psi)$, then for all $A \in \mathcal{P}(\Re^m)$ it follows that $\psi(A) = \bigcup\{x \in \Re^m : A - x \in \mathcal{V}(\psi)\}$. If $\mathcal{V} \subset \mathcal{P}(\Re^m)$, then the operator φ defined on $\mathcal{P}(\Re^m)$ by $\varphi(A) = \bigcup\{x \in \Re^m : A - x \in \mathcal{V}\}$ $(A \in \mathcal{P}(\Re^m))$ is $\mathbf{R}_m$-invariant and the kernel of φ is $\mathcal{V}$.*

Proof The first part is simply the straightfoward transcription of the general result $\psi(x) = \bigvee\{\eta \in \ell : \tau_{-\eta}(x) \in \mathcal{V}(\psi)\}$ to the case at hand. The rest is Proposition 30, with the difference that $\mathcal{V}(\varphi) = \mathcal{V}$ rather than $\mathcal{V}(\varphi) \supset \mathcal{V}$. To prove the latter we simply evaluate

$$\mathcal{V}(\varphi) = \{A \in \mathcal{P}(\Re^m) : \mathbf{0} \in \varphi(A)\} = \{A \in \mathcal{P}(\Re^m) : \mathbf{0} \in \bigcup\{x \in \Re^m : A - x \in \mathcal{V}\}\}.$$

Thus $A \in \mathcal{V}(\varphi)$ if and only if $A = A - \mathbf{0} \in \mathcal{V}$.

Thus, in the case of this example, the lattice isomorphism $\psi \longmapsto \mathcal{V}(\psi)$ of $\mathcal{O}_G$ onto $\mathcal{K}(\mathcal{L})$ is actually onto the lattice of all subsets of $\mathcal{P}(\Re^m)$, i.e., every subset of $\mathcal{P}(\Re^m)$ is a kernel set, and the complete lattice of $\mathbf{R}_m$-invariant operators on $\mathcal{P}(\Re^m)$ is isomorphic to the complete lattice $(\mathcal{P}(\mathcal{P}(\Re^m)), \cap, \cup)$.

Remark 33 *If $B, C \in \mathcal{P}(\Re^m)$, then the operators ε_B (erosion by B), $\tilde{\delta}_C$, and σ_{BC} are given for all $A \in \mathcal{P}(\Re^m)$ by the following.*

1. $\varepsilon_B(A) = \bigcup\{x \in \Re^m : B \subset A - x\}$.
2. $\tilde{\delta}_C(A) = \bigcup\{x \in \Re^m : A - x \subset C\}$.
3. $\sigma_{BC}(A) = \bigcup\{x \in \Re^m : B \subset A - x \subset C\}$.

Proposition 44 *If $B, C \in \mathcal{P}(\Re^m)$, then we have the following.*

1. $\mathcal{V}(\varepsilon_B) = \{A \in \mathcal{P}(\Re^m) : B \subset A\} = [B, \Re^m]$.
2. $\mathcal{V}(\widetilde{\delta}_C) = \{A \in \mathcal{P}(\Re^m) : A \subset C\} = [\emptyset, C]$.
3. $\mathcal{V}(\sigma_{BC}) = \{A \in \mathcal{P}(\Re^m) : B \subset A \subset C\} = [B, C] = \mathcal{V}(\varepsilon_B) \cap \mathcal{V}(\widetilde{\delta}_C)$.
4. *If $B \not\subset C$, then $\sigma_{BC}(A) = \emptyset$ for all $A \in \mathcal{P}(\Re^m)$ and $\mathcal{V}(\sigma_{BC}) = \emptyset$.*

Proof For (1) we have $\mathcal{V}(\varepsilon_B) = \{A : \mathbf{0} \in \bigcup\{x \in \Re^m : B \subset A - x\}\}$. Thus $A \in \mathcal{V}(\varepsilon_B)$ if and only if $B \subset A - \mathbf{0} = A$. (2) and (3) follow in the same obvious manner, and (4) is simply the transcription of the general result of Remark 13.

Corollary 24 *If $B, C \in \mathcal{P}(\Re^m)$, then $\sigma_{BC} = \varepsilon_B \wedge \widetilde{\delta}_C$.*

Lemma 22 *If $C \in \mathcal{P}(\Re^m)$, then $\widetilde{\delta}_C$ is an anti-dilation on $\mathcal{P}(\Re^m)$.*

Proof Let $\mathcal{A}$ be an arbitrary subset of $\mathcal{P}(\Re^m)$. Then $\widetilde{\delta}_C(\bigcup \mathcal{A}) = \bigcup\{x \in \Re^m : \bigcup \mathcal{A} - x \subset C\}$. Thus $\widetilde{\delta}_C(\bigcup \mathcal{A})$ is the set of points x such that the translate $A - x$ of every $A \in \mathcal{A}$ is contained in C. Consequently,

$$\widetilde{\delta}_C(\bigcup \mathcal{A}) = \bigcap\{\bigcup\{x \in \Re^m : A - x \subset C\} : A \in \mathcal{A}\} = \bigcap\{\widetilde{\delta}_C(A) : A \in \mathcal{A}\}.$$

In view of this result, we call $\widetilde{\delta}_C$ anti-dilation by C.

Remark 34 *If $x \in \Re^m$, then the only element of ℓ that $\{x\}$ is $\subset$-comparable with is $\{x\}$.*

4.3 Example 3: $\mathcal{L}_1 = \mathrm{ERV}(\Re^n)$ and $\mathcal{L}_2 = \mathcal{P}(\Re^{n+1})$

Here is the first example in which the domain and range lattices are distinct. In this case, the sup-generating subset ℓ of $\mathcal{L}_2$ is the set of singletons $\{(x, t) : x \in \Re^n, t \in \Re\}$ of $\Re^{n+1}$, $\mathbf{R}_{n+1}$ is the group $(\Re^{n+1}, +)$, σ is given by

$$\sigma((x,t), A) = A + (x,t) \equiv \{(a + x, \tau + t) : (a, \tau) \in \mathcal{A}\},$$

and s is given by $s((x,t), f) = f + (x,t)$. As in Example 2, $(\mathbf{R}_{n+1}, \sigma)$ acts effectively as a group of automorphisms on $\mathcal{L}_2$ and also acts ℓ-admissably on $\mathcal{L}_2$. As in Example 1, $\mathcal{X} \equiv \{\eta_{x,t} : x \in \Re^n, t \in \Re\}$ is a sup-generating subset of $\mathcal{L}_1$, $(\mathbf{R}_{n+1}, s)$ acts effectively as a group of automorphisms on $\mathcal{L}_1$, and $(\mathbf{R}_{n+1}, s)$ acts $\mathcal{X}$-admissably on $\mathcal{L}_1$.

We choose $r = \eta_{0,0}$ and $\rho = (\mathbf{0}, 0)$ as the reference elements of $\mathcal{X}$ and ℓ, respectively, and obtain the bijections $\xi \longmapsto \tau_\xi$ and $\Xi \longmapsto T_\Xi$ of ℓ and $\mathcal{X}$, respectively, onto the automorphism groups $(\{\sigma_{(x,t)} : (x,t) \in \mathbf{R}_{n+1}\}, \circ)$ and $(\{s_{(x,t)} : (x,t) \in \mathbf{R}_{n+1}\}, \circ)$; here, $\tau_\xi = \sigma_{g_{\rho\xi}}$ and $T_\Xi = s_{g_{r\Xi}}$. Moreover, the natural bijection $\xi \longmapsto \Xi$ of ℓ onto $\mathcal{X}$ given by $(x,t) \longmapsto \eta_{x,t}$ is such that $g_{\rho(x,t)} = g_{r\eta_{x,t}}$, i.e., $\theta_{(x,t)} = T_{\eta_{x,t}}$, for all $(x,t) \in \mathbf{R}_{n+1}$. We note the following:

(A) $(\ell, \preceq)$ and $(\mathcal{X}, \preceq)$ are poset isomorphic via $(x,t) \longmapsto \eta_{x,t}$.

(B) $(\ell, \preceq)$ and $(\mathbf{T}, \preceq_p)$ are poset isomorphic via $(x,t) \longmapsto \tau_{(x,t)}$.

(C) $(\mathcal{X}, \preceq)$ and $(\{T_{\eta_{x,t}} : (x,t) \in \mathbf{R}_{n+1}\}, \preceq_p)$ are poset isomorphic via $\eta_{x,t} \longmapsto T_{\eta_{x,t}}$.

(D) $(\{T_{\eta_{x,t}} : (x,t) \in \mathbf{R}_{n+1}\}, \preceq_p)$ and $(\mathbf{\Theta}, \preceq_p)$ are poset isomorphic via $T_{\eta_{x,t}} \longmapsto \theta_{(x,t)}$.

Thus we see that $(\mathbf{\Theta}, \preceq_p)$ and $(\mathbf{T}, \preceq_p)$ are poset isomorphic via $\theta_{(x,t)} \longmapsto \tau_{(x,t)}$.

Remark 35 *A mapping $\psi : \mathrm{ERV}(\Re^n) \longrightarrow \mathcal{P}(\Re^{n+1})$ is $(\mathbf{R}_{n+1}, \sigma)$-invariant (or more briefly $\mathbf{R}_{n+1}$-invariant) if and only if $\psi(f+(x,t)) = \psi(f)+(x,t)\ \forall\ f \in \mathrm{ERV}(\Re^n)$ and $(x,t) \in \mathbf{R}_{n+1}$.*

Accordingly, if $\psi : \mathrm{ERV}(\Re^n) \longrightarrow \mathcal{P}(\Re^{n+1})$ is $\mathbf{R}_{n+1}$-invariant, then

$$\mathcal{V}(\psi) = \{f \in \mathrm{ERV}(\Re^n) : (\mathbf{0}, 0) \subset \psi(f)\}.$$

Proposition 45 *If $\psi : \mathrm{ERV}(\Re^n) \longrightarrow \mathcal{P}(\Re^{n+1})$ is $\mathbf{R}_{n+1}$-invariant with kernel $\mathcal{V}(\psi)$, then for all $f \in \mathrm{ERV}(\Re^n)$ it follows that $\psi(f) = \bigcup\{(x,t) \in \Re^{n+1} : f - (x,t) \in \mathcal{V}(\psi)\}$. If $\mathcal{V} \subset \mathrm{ERV}(\Re^n)$, then the mapping φ defined on $\mathrm{ERV}(\Re^n)$ by*

$$\varphi(f) = \bigcup\{(x,t) \in \Re^{n+1} : f - (x,t) \in \mathcal{V}\}$$

is $\mathbf{R}_{n+1}$-invariant and the kernel of φ is $\mathcal{V}$.

Thus, in the case of this example, the lattice isomorphism $\psi \longmapsto \mathcal{V}(\psi)$ of $\mathcal{O}_{12}^G$ onto $\mathcal{K}_1$ is actually onto the lattice of all subsets of $\mathrm{ERV}(\Re^n)$, i.e., every subset of $\mathrm{ERV}(\Re^n)$ is a kernel set, and the complete lattice of $\mathbf{R}_{n+1}$-invariant maps of $\mathrm{ERV}(\Re^n)$ into $\mathcal{P}(\Re^{n+1})$ is isomorphic to the complete lattice $(\mathcal{P}(\mathrm{ERV}(\Re^n)), \cap, \cup)$.

Remark 36 *If $f, g, h \in \mathrm{ERV}(\Re^n)$, then we have the following.*

1. *$\varepsilon_g(f) = \bigcup\{(x,t) : g \preceq f - (x,t)\}$ and $\mathcal{V}(\varepsilon_g) = [g, \aleph_{\Re^n}]$.*
2. *$\widetilde{\delta}_h(f) = \bigcup\{(x,t) : f - (x,t) \preceq h\}$ and $\mathcal{V}(\widetilde{\delta}_h) = [\aleph_\emptyset, h]$.*
3. *$\sigma_{gh}(f) = \bigcup\{(x,t) : g \preceq f - (x,t) \preceq h\}$ and $\mathcal{V}(\sigma_{gh}) = [g, h]$.*
4. *ε_g is an erosion, $\widetilde{\delta}_h$ is an anti-dilation, and $\sigma_{gh} = \varepsilon_g \wedge \widetilde{\delta}_h$ is sup-generating.*

If $h \in \mathrm{ERV}(\Re^n)$, then $\widetilde{\delta}_h$ is called anti-dilation by h.

Remark 37 *If $\psi : \mathrm{ERV}(\Re^n) \longrightarrow \mathcal{P}(\Re^{n+1})$ is $\mathbf{R}_{n+1}$-invariant, then the Ronse decomposition $\psi = \sup\{\sigma_{gh} : [g,h] \subset \mathcal{V}(\psi)\}$ can be written $\psi = \sup\{\varepsilon_g \wedge \widetilde{\delta}_h : [g,h] \subset \mathcal{V}(\psi)\}$, and σ_{gh} is a sup-generating mapping for all $g, h \in \mathrm{ERV}(\Re^n)$.*

This completes the illustrative examples.

5 Closed Kernel Conjecture

For the definitions of and theoretical interrelations among a number of the concepts used in this section, the reader will have to have recourse to the previous work [3]; indeed, to include such material here would considerably lengthen this already very long report.

5.1 Preliminary Considerations

Definition 49 *Let X be a topological space, let G be a topological group, let $\mathcal{H} : g \longmapsto \sigma_g$ be a homomorphism of G onto a group $(\{\sigma_g : g \in G\}, \circ)$ of transformations of X, and let the mapping $\sigma : G \times X \longrightarrow X$ be defined by $\sigma(g, x) = \sigma_g(x)$. The group action (G, σ) thusly defined on X is called a* **continuous action** *if σ is a continuous map.*

Remark 38 *Note the following facts.*

1. *If X is a topological space, G is a topological group, and (G, σ) is a continuous group action on X, then σ_g is a homeomorphism of X for all $g \in G$.*
2. *If $\mathcal{L}$* **is a lattice with a topology**, *if G is a topological group, and if (G, σ)* **acts continuously and effectively on $\mathcal{L}$ as a group of lattice automorphisms**, *then each σ_g is both a lattice isomorphism and a homeomorphism of $\mathcal{L}$, i.e.,* **each σ_g is an automorphism of $\mathcal{L}$ relative to both its lattice and topological structure.**

Our interest in this topological addition lies in the situation of the above remark when G is abelian and $\mathcal{L}$ is a UC lattice [3, Def. 9] with the topology $\mathbf{m}(\mathcal{L})$ [3, Def. 13]. Let us formalize and elaborate on this situation.

Definition 50 *If $\mathcal{L}$ is a UC lattice with the topology $\mathbf{m}(\mathcal{L})$, if G is an abelian topological group, and if (G, σ) acts continuously and effectively as a group of automorphisms on $\mathcal{L}$, then we will say that (G, σ)* **acts m-admissably on $\mathcal{L}$.**
If $\mathcal{L}$ additionally has a sup-generating subset ℓ, and if (G, σ) also acts ℓ-admissably on $\mathcal{L}$, then we will say that (G, σ) **acts mℓ-admissably on $\mathcal{L}$.**

Recall that if (G, σ) acts ℓ-admissably, then (G, σ) is a transitive-regular action on ℓ, i.e., for each $(\xi, \eta) \in \ell \times \ell$ there is exactly one $g_{\xi\eta} \in G$ such that $\sigma_{g_{\xi\eta}}(\xi) = \eta$. In fact, for each fixed $\xi \in \ell$ the mapping $\eta \longmapsto g_{\xi\eta}$ is a bijection of ℓ onto G. In the case of $\mathbf{m}\ell$-admissability, since ℓ has its relative $\mathbf{M}$-topology [3, Def. 13] and G has its topological group topology, we may further inquire as to the continuity properties of the map $(\xi, \eta) \longmapsto g_{\xi\eta}$ of $\ell \times \ell$ onto G.

Definition 51 *Let $\mathcal{L}$ be a UC lattice with a sup-generating subset ℓ, let G be an abelian topological group, and let (G, σ) act $\mathbf{m}\ell$-admissably on $\mathcal{L}$. If $(\xi, \eta) \longmapsto g_{\xi\eta}$ is a continuous map of $\ell \times \ell$ onto G, then (G, σ) will be called* **a totally mℓ-admissable action on $\mathcal{L}$.**

Proposition 46 *If (G, σ) is a totally $\mathbf{m}\ell$-admissable action on a UC lattice $\mathcal{L}$, then for each fixed $\xi \in \ell$ the bijection $\eta \longmapsto g_{\xi\eta}$ is a homeomorphism of ℓ onto G.*

Proof It is clear that $\eta \longmapsto g_{\xi\eta}$ is a continuous bijection of ℓ onto G for each fixed $\xi \in \ell$. The inverse of $\eta \longmapsto g_{\xi\eta}$ is given by $g \longmapsto \sigma(g, \xi)$. Since σ is continuous on $G \times \mathcal{L}$, it is clear that $g \longmapsto \sigma(g, \xi)$ is continuous on G for each fixed $\xi \in \ell$.

With a totally $\mathbf{m}\ell$-admissable action, then, we have the family $\{\mathcal{I}_\xi : \xi \in \ell\}$ of topological-group isomorphisms between $(\ell, +)$ and $(G, \cdot)$ given by $\mathcal{I}_\xi : \eta \longmapsto g_{\xi\eta}$.

5.2 Statement of Conjecture and its Motivation

For the conjecture, we assume the following:

1. $\mathcal{L}_1$ is a UC lattice with a sup-generating subset $\mathcal{X}$, $\mathcal{L}_2$ is a complete lattice with a meet-complete UC sublattice $\mathcal{F}$, and $\mathcal{L}_2$ has a sup-generating subset $\ell \subset \mathcal{F}$.

2. $\mathbf{M}(\mathcal{L}_1)$ and $\mathbf{M}(\mathcal{F})$ are non-bicontinuous [4] Matheron spaces [3, Def. 13, sect. 3.4].

3. There is a complete continuous homomorphism $\mathbf{\Phi}$ of $\mathcal{F}$ onto $\mathcal{L}_1$.

4. Let $\gg$ denote the **way above** relation [3, Def. 7] in either $\mathcal{L}_1$ or $\mathcal{F}$. Then $\eta \not\gg \eta$ for all $\eta \in \mathcal{X}$ and $\xi \not\gg \xi$ for all $\xi \in \ell$.

5. G is an abelian topological group that acts effectively as a group of automorphisms on both $\mathcal{L}_1$ and $\mathcal{L}_2$ by means of the maps s and σ.

6. (G, s) acts in a totally $\mathbf{m}\ell$-admissable fashion on $\mathcal{L}_1$.

7. (G, σ) acts $\mathcal{X}$-admissably on $\mathcal{L}_2$, and acts in a totally $\mathbf{m}\mathcal{X}$-admissable fashion on $\mathcal{F}$.

Closed Kernel Conjecture. *A G-invariant mapping $\psi : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is into $\mathcal{F}$ and USC if and only if $\mathcal{V}(\psi)$ is a closed subset of $\mathcal{L}_1$.*

The motivation for this conjecture lies in the following two of its special cases.

Theorem 11 *Let $\mathbf{F}$ denote the complete lattice of closed subsets of $\Re^n$ relative to the meet and join operations $\cap$ and $\cup$, and let $\mathcal{P}$ denote the complete lattice of all subsets of $\Re^n$, likewise relative to the meet and join operations $\cap$ and $\cup$. Let $\psi : \mathbf{F} \longrightarrow \mathcal{P}$ be translationally invariant in the sense that*

$$\psi(F + x) = \psi(F) + x$$

for all $F \in \mathbf{F}$ and all $x \in \Re^n$. Then ψ is into $\mathbf{F}$ and USC if and only if the kernel of ψ is closed in $\mathbf{F}$ relative to the hit-miss topology of $\mathbf{F}$.

The setting of this "closed kernel theorem," which in essence was proved by Matheron [12], is closed Euclidean set morphology. The context of the second special "closed kernel theorem," which was proved in [2], is upper semicontinuous function morphology; it reads as follows.

Theorem 12 *Let* $\mathrm{ERV}(\Re^n)$ *denote the complete lattice of extended real valued functions on* $\Re^n$ *relative to the meet and join operations given by the pointwise infimum and supremum, and let* $\mathrm{USC}(\Re^n)$ *denote the complete lattice of upper semicontinuous functions in* $\mathrm{ERV}(\Re^n)$, *likewise relative to the meet and join operations given by the pointwise infimum and supremum. Let* $\psi : \mathrm{USC}(\Re^n) \longrightarrow \mathrm{ERV}(\Re^n)$ *be translationally invariant in the sense that*

$$\psi(f + (x,t)) = \psi(f) + (x,t) \text{ for all } f \in \mathrm{USC}(\Re^n) \text{ and all } (x,t) \in \Re^n \times \Re,$$

where $g + (x,t)$ *is defined for all* $g \in \mathrm{ERV}(\Re^n)$ *and all* $(x,t) \in \Re^n \times \Re$ *by* $(g + (x,t))(y) = g(y - x) + t$. *Then* ψ *is into* $\mathrm{USC}(\Re^n)$ *and USC if and only if the kernel of* ψ *is closed in* $\mathrm{USC}(\Re^n)$ *relative to the* $\mathbf{M}$*-topology of* $\mathrm{USC}(\Re^n)$.

It is now, finally, time to conclude.

6 Conclusion

The Banon-Barrera theory of complete-lattice mappings has succeeded in further and substantially elaborating the developments of Serra's school [5], which implemented its view that **mathematical morphology is essentially a theory of mappings of one complete lattice into another,** that image universes are most generally modeled by complete lattices and that the morphologically useful transformations of images are mappings of one such image universe into itself or perhaps another. As we have seen, the Banon-Barrera theory develops pure lattice theoretical characterizations of the basic morphological mappings of erosion, dilation, etc, and shows how all complete-lattice mappings can be represented lattice algebraically in terms of these basic morphological mappings. Their decomposition theorem and their concept of a morphological connection are very general and powerful theoretical tools for the investigation of morphological questions that can be formulated lattice algebraically. As we have also seen, the work of Heijmanns and Ronse has made this lattice algebraic view more concrete and intuitive by postulating the additional structure of a sup-generating subset together with a compatible abelian subgroup G of the lattice's automorphism group in order to abstractly model the translation-invariance present in concrete morphology theories, and have thereby laid the groundwork for the investigation of other possible morphological symmetries. **For Heijmanns and Ronse, mathematical morphology is in essence a theory of G-invariant complete-lattice mappings.**

This report has investigated the question of how G-invariant complete-lattice maps behave in contradistinction to general complete-lattice maps, and has more specifically determined the extent to which the general theory of Banon and Barrera is reproduced in the restricted realm of G-invariant complete-lattice maps. By deriving, wherever possible and appropriate, the G-invariant versions of the general theory's results, I have shown that most of the Banon-Barrera theory is indeed reproduced for G-invariant maps, and have moreover shown how and in what form it is reproduced. In this way, the two theories have been joined together to their mutual benefit and further developed. This accomplishment, together with the conjecture I have presented—assuming that it turns out to be true, either as stated or in some modified form—has come close to realizing the previously stated goal [3] of achieving an abstract mathematical system that exhibits most of the algebraic and topological properties of the more concrete morphologies, one that is consequently a much more general morphological theory with a considerably wider range of application, and so one that should make new and more effective applications to ATR and computer vision problems possible. Apart from resolving the status of the "closed kernel conjecture" and elaborating the implications that arise therefrom, the remaining element needed to fully reach the stated goal is the generalization of the probabilistic aspect of the standard morphologies to the mentioned abstract mathematical system, i.e., the effective generalization of Matheron's concept of a *random closed set* to that of a random variable in an $\mathbf{M}$-topologized upper continuous lattice.

7 References

1. D. W. McGuire, *The Morphological Processing of Binary Images*, Army Research Laboratory, ARL-TR-28 (1993).

2. D. W. McGuire, *Greyscale Morphology by the Umbra Method*, Army Research Laboratory, ARL-TR-595 (1994).

3. D. W. McGuire, *Continuous Lattices and Mathematical Morphology*, Army Research Laboratory, ARL-TR-1548 (1998).

4. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin (1980).

5. J. Serra (Editor), *Image Analysis and Mathematical Morphology Volume II: Theoretical Advances*, Academic Press (1988).

6. H.J.A.M. Heijmans and C. Ronse, *The Algebraic Basis of Mathematical Morphology I. Dilations and Erosions*, CVGIP, 50, 245–295 (1990).

7. H.J.A.M. Heijmans, *Morphological Image Operators*, Academic Press (1994).

8. G.J.F. Banon and J. Barrera, *Decomposition of Mappings between Complete Lattices by Mathematical Morphology, Part I. General lattices*, Signal Processing, Elsevier, 30, 299–327 (1993).

9. J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press (1982).

10. G. Birkhoff, *Lattice Theory*, 3rd. ed., Vol. 25, Amer. Math. Soc. Colloq. Pubs. (1984).

11. G.J.F. Banon and J. Barrera, *Minimal Representations for Translation-Invariant Set Mappings by Mathematical Morphology*, SIAM J. Appl. Math., **51**, 6, 1782–1798 (1991).

12. G. Matheron, *Random Sets and Integral Geometry*, John Wiley & Sons (1975).

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The relations between two abstract lattice-algebraical approaches to mathematical morphology are investigated. One approach, developed by Heijmans and Ronse, entails the use of an abelian automorphism group, *G*, acting transitively on a sup-generating subset of the lattice, in order to abstract the translation invariance present in concrete morphology theories. The other, developed by Banon and Barrera, analyzes general mappings between complete lattices and develops morphological decomposition formulas for such mappings. By determining the *G*-invariant forms of the concepts and theorems of the Banon-Barrera theory, the present investigation combines the two theories into a coherent whole and develops them further.

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Mathematical morphology, lattice theory, complete-lattice mappings, dilation, erosion, adjunction, morphological connection, Galois connection, group-invariant lattice mappings, mapping decomposition, kernel, image understanding, image recognition, image processing

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